Exploring the Theory of Geometric Bifurcation

Jay H. Wolkowisky
Department of Mathematics
University of Colorado
Boulder, Colorado 80309, USA
wolkowis@euclid.colorado.edu

Abstract

This paper will deal with the Theory of Geometric Bifurcation which the author developed in 1986. The theory developed in those papers is very general and abstract. So, until a symbolic software program such as *Mathematica* came along, it was very difficult to examine concrete examples which would illustrate and explain the theory. In this paper we will look at examples of bifurcating branches of solutions of nonlinear: algebraic equations, ordinary differential equations, and partial differential equations.

1. Introduction

This paper will investigate nonlinear eigenvalue problems of the form
Lw - w + F(,w) = 0, where **L** is a linear operator in a Hilbert space **H**, with being a scalar parameter, and **F(,w) = o(||w||)**. Such equations arise in bifurcation (branching) problems where a solution becomes non-unique and bifurcates into more than one solution. For example, in the buckling of rods, plates, and shells. Also in the field of fluid dynamics, such as the Bernard and Taylor problems.

In this paper we will look at examples of the following types:
(a) **L = A**, **A** is a n x n matrix,
(b) $\mathbf{L} = \frac{\mathbf{d}^2}{\mathbf{dx}^2}$,
(c) $\mathbf{L} = \nabla^2$, the Laplace operator.

We will investigate the existence(or non-existence) of global branches of solutions of the above equations. We also will show how to construct these branches. The method used to accomplish this is based on the"Theory of Geometric Bifurcation" developed by the author [**1**,**2**]. The above equation has **w 0** as a solution for all values of **.** The "linearized" equation **Lw - w = 0**, besides having this **trivial solution** also has non-trivial solutions at the eigenvalues of **L**, = $_i$. The main question we will be interested in is whether the nonlinear equation **Lw - w + F(,w) = 0** has non-trivial solutions bifurcating from the trivial solution at the eigenvalues = $_i$. (It is easy to show that if the nonlinear equation has non-trivial solutions they must bifurcate from the trivial solution at the eigenvalues of the linearized equation). We will be restricting our attention to the case that the operator **L** is self-adjoint (symmetric), which is the case for most physical models. This will allow the results proved in [**1**,**2**] to be stated more simply. We will use **dim** and **null** for dimensiom and null space respectively. When **dim null**(L - $_i$I) = 1 (**multiplicity of** $_i$ **= 1**), then bifurcation from the eigenvalue $_i$ always occurs. This case is very simple to handle and not very interesting. We will only be looking at the case when the **multiplicity of** $_i$ **is 2**. If we let

w = (y +), (1)

where

y N= null(L - $_0$I),
||y|| = 1,

and

N$^\perp$ (the orthogonal compliment of **N, H = N N**$^\perp$).

Where is a small parameter, and $_0$ is an eigenvalue of **L** with

dim N 2.

We can now state the above mentioned results. But first we define **T**[y], which is the projection operator of **H** onto the tangent plane to the unit sphere in **N** at y, i.e.,

T[y] **: H** [y] = {x | ⟨x , y⟩ = 0, x and y **N**, ||y|| = 1},
with ⟨ , ⟩ being the inner product on **H**.

<u>**Theorem 1**</u>

$_0$ *is a bifurcation point of*

Lw - w + F(,w) = 0 (2)

if and only if for each > 0 *sufficiently small there exists a* y **N** *with* ||y|| = 1 *such that it satisfies*

T[y]F(^(y,), (y + ^(y,))) = 0. (3)

Where $\hat{\ }(y, \ )$ *and* $\hat{\ }(y, \ )$ *are solutions of the equations*

$$L - {}_0 - (- {}_0)(y + \ ) + {}^1 F(\ , \ (y + \ )) = 0 \qquad (4)$$

$$- {}_0 - \langle {}^1 F(\ , \ (y + \ )), y\rangle = 0, \qquad (5)$$

for a fixed y *and* .

Loosely speaking, this theorem says that solutions of (2) will bifurcate from the zero solution at an eigenvalue $_0$ of L if and only if the tangential vector field
T[y]F($\hat{\ }$(y,), (y + $\hat{\ }$(y,))) has a zero for each sufficiently small .

It can be shown that $\hat{\ }$(y,0) = $_0$ and $\hat{\ }$(y,0) = 0. In addition, if we assume that

F(, w) = F(,w), with >1,

then the leading term in approximating Equation (3) is

$$\mathbf{T}[y]F({}_0,y) = 0 . \qquad (6)$$

These ideas are illustrated in Figure 1.

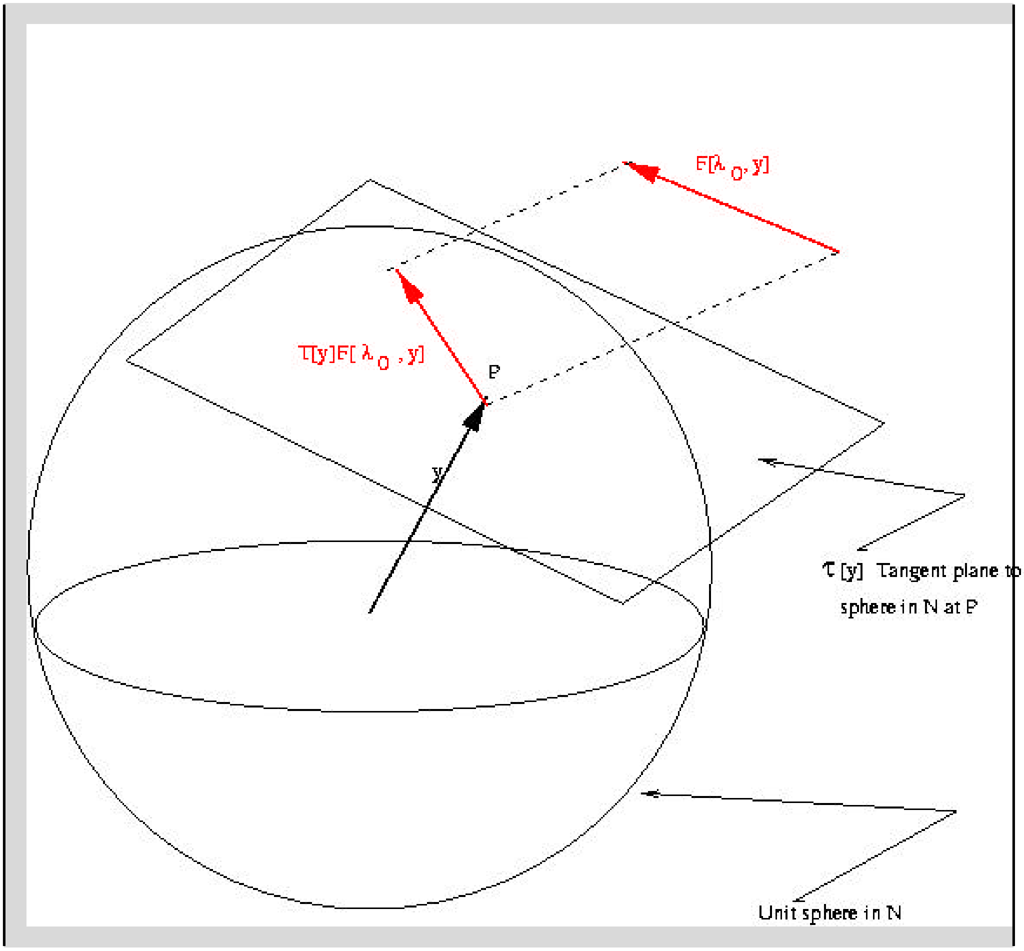


Figure 1

If this tangential vector field has a **"transverse zero"**, then bifurcation from $_0$ will occur. A **transverse zero** of a vector field is a zero of the vector field such that any small perturbation of the vector field will not eliminate the zero. For example, in one dimension x^3 has a transverse zero at x = 0, while x^2 does not.

It is shown in [**1**] that if the multiplicity of an eigenvalue $_0$ is **odd**, then bifurcation from $_0$ will occur. In [**2**] it is shown that if F(,w) is an **<u>even</u>** function of the variable w, then bifurcation from $_0$ will also occur. An even function being one that satisfies the condition F(,-w) = F(,w). **It is for these reasons that, in this paper, we will be looking at examples where the multiplicity of $_0$ is <u>even</u> and F(,w) is an <u>odd</u> function of** w. **In this case bifurcation from $_0$ may or may not occur, as the following examples will show.**

2. L = A, an n ×n matrix and H = R^n .

We first consider the following example for **n=4.**

$$A \quad \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix};$$

$$w \quad \begin{pmatrix} x \\ y \\ u \\ v \end{pmatrix}$$

$$\mathrm{F}[\ ,w] \quad \begin{pmatrix} x^3 \quad y^3 \quad u\,y^2 \quad v\,y^2 \\ x^3 \quad v\,x^2 \quad y^2\,x \quad u\,y\,x \\ x^3 \quad u\,y\,x \quad y^3 \quad v\,y^2 \\ x^3 \quad u\,y\,x \quad y^3 \quad v\,y^2 \quad u\,v\,y \end{pmatrix};$$

So that equation (2) becomes

```
A.w – λ w + F[λ, w] == 0
```

$$\begin{pmatrix} x^3 \quad x \quad y^3 \quad u\,y^2 \quad v\,y^2 \quad x \\ x^3 \quad v\,x^2 \quad y^2\,x \quad u\,y\,x \quad y \quad y \\ x^3 \quad u\,y\,x \quad y^3 \quad v\,y^2 \quad 2\,u \quad u \\ x^3 \quad u\,y\,x \quad y^3 \quad v\,y^2 \quad 2\,v \quad u\,v\,y \quad v \end{pmatrix} \quad 0$$

The eigenvalues of **A** are **{1, 1, 2, 2}**. We will be investigating the bifurcation from the double eigenvalue

$$\lambda_0 = 1$$

Therefore **dim N** = **dim N$^\perp$** = **2**.

In this case, bases for **N** and **N$^\perp$** are $\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \right\}$ and $\left\{ \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}$ respectively.

The vectors in the unit sphere(circle) in **N** can then be parametrized using the angle in **[0 , 2)** by

$$\mathbf{y} = \begin{pmatrix} \mathbf{Cos}[\alpha] \\ \mathbf{Sin}[\alpha] \\ 0 \\ 0 \end{pmatrix}$$

and the vectors in **N**$^{\perp}$ by

$$\eta = \begin{pmatrix} 0 \\ 0 \\ \eta 1 \\ \eta 2 \end{pmatrix}$$

So that (1) becomes,

$$\mathbf{w} = \epsilon \begin{pmatrix} \mathbf{Cos}[\alpha] \\ \mathbf{Sin}[\alpha] \\ \eta 1 \\ \eta 2 \end{pmatrix}$$

Equation (2) can then be written as

$$\left(\mathbf{A}. \begin{pmatrix} \epsilon\, \mathbf{Cos}[\alpha] \\ \epsilon\, \mathbf{Sin}[\alpha] \\ \epsilon\, \eta 1 \\ \epsilon\, \eta 2 \end{pmatrix} - \lambda \begin{pmatrix} \epsilon\, \mathbf{Cos}[\alpha] \\ \epsilon\, \mathbf{Sin}[\alpha] \\ \epsilon\, \eta 1 \\ \epsilon\, \eta 2 \end{pmatrix} + \begin{pmatrix} x^3 + y^3 + u\, y^2 + v\, y^2 \\ x^3 + v\, x^2 + y^2\, x + u\, y\, x \\ x^3 + u\, y\, x + y^3 + v\, y^2 \\ x^3 + u\, y\, x + y^3 + v\, y^2 + u\, v\, y \end{pmatrix} /. \{x \to \epsilon\, \mathbf{Cos}[\alpha],\ y \to \epsilon\, \mathbf{Sin}[\alpha],\ u \to \epsilon\, \eta 1,\ v \to \epsilon\, \eta 2\}\right) == \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Since $\begin{pmatrix} \mathbf{Sin}[\alpha] \\ -\mathbf{Cos}[\alpha] \\ 0 \\ 0 \end{pmatrix}$ is orthogonal to $\begin{pmatrix} \mathbf{Cos}[\alpha] \\ \mathbf{Sin}[\alpha] \\ 0 \\ 0 \end{pmatrix}$ then Equation (3) becomes

$$\mathbf{Dot}\left[\mathbf{Flatten}\left[\begin{pmatrix} \mathbf{Sin}[\alpha] \\ -\mathbf{Cos}[\alpha] \\ 0 \\ 0 \end{pmatrix}\right], \mathbf{Flatten}\left[\begin{pmatrix} x^3 + y^3 + u\, y^2 + v\, y^2 \\ x^3 + v\, x^2 + y^2\, x + u\, y\, x \\ x^3 + u\, y\, x + y^3 + v\, y^2 \\ x^3 + u\, y\, x + y^3 + v\, y^2 + u\, v\, y \end{pmatrix} /. \{x \to \epsilon\, \mathbf{Cos}[\alpha],\ y \to \epsilon\, \mathbf{Sin}[\alpha],\ u \to \epsilon\, \eta 1,\ v \to \epsilon\, \eta 2\}\right]\right] == 0$$

```
(*7*)
```

The leading term of this equation, which is Equation (6), obtained by letting **1=0** and **2=0** , becomes

```
Simplify[% /. {η1 -> 0, η2 -> 0}]
```

In order to calculate the zeros of this equation, let

```
t[α_] = -1 - 8 Cos[2 α] + Cos[4 α] + 2 Sin[2 α] + Sin[4 α];
```

and

```
Plot[t[α], {α, 0, 2 π}]
```

This shows, using the above Theorem and the remark about **transverse zeros** , that bifurcation will occur for **=1** at each of the four zeros of **t[ ]**. We can calculate these zeros by

```
α1 = α /. FindRoot[t[α], {α, .8}]
```

```
α2 = α /. FindRoot[t[α], {α, 2.2}]
```

```
α3 = α /. FindRoot[t[α], {α, 3.8}]
```

```
α4 = α /. FindRoot[t[α], {α, 5.3}]
```

We note that **1 - 3 =** and **2 - 4 =** , which means that the solution branches bifurcate in pairs in opposite directions. This is expected since the **nonlinearity is cubic**, so that if `(x,y,u,v)` is a solution then `(-x,-y,-u,-v)` is also a solution.

Now that we know bifurcation will occur, we can construct the bifurcating branches.

■ 2.1 Constructing the Solution Branches

We first find **1** and **2** for small .

Equation (4) can be written, using, $_0$ **1**, as

$$\left(A.\begin{pmatrix}0\\0\\\eta 1\\\eta 2\end{pmatrix} - \begin{pmatrix}0\\0\\\eta 1\\\eta 2\end{pmatrix} - (\lambda - 1)\begin{pmatrix}\text{Cos}[\alpha]\\\text{Sin}[\alpha]\\\eta 1\\\eta 2\end{pmatrix} + \frac{1}{\epsilon}\begin{pmatrix}x^3 + y^3 + u\,y^2 + v\,y^2\\x^3 + v\,x^2 + y^2\,x + u\,y\,x\\x^3 + u\,y\,x + y^3 + v\,y^2\\x^3 + u\,y\,x + y^3 + v\,y^2 + u\,v\,y\end{pmatrix} /.\ \{x \to \epsilon\,\text{Cos}[\alpha],\ y \to \epsilon\,\text{Sin}[\alpha],\ u \to \epsilon\,\eta 1,\ v \to \epsilon\,\eta 2\}\right) == \begin{pmatrix}0\\0\\0\\0\end{pmatrix} \quad (*8*)$$

The first two elements in the above vector equation are

$$\text{Cos}[\alpha]^3\,\epsilon^2 + \text{Sin}[\alpha]^3\,\epsilon^2 + \eta 1\,\text{Sin}[\alpha]^2\,\epsilon^2 + \eta 2\,\text{Sin}[\alpha]^2\,\epsilon^2 - (\lambda - 1)\,\text{Cos}[\alpha] == 0$$

and

$$\text{Cos}[\alpha]^3\,\epsilon^2 + \eta 2\,\text{Cos}[\alpha]^2\,\epsilon^2 + \text{Cos}[\alpha]\,\text{Sin}[\alpha]^2\,\epsilon^2 + \eta 1\,\text{Cos}[\alpha]\,\text{Sin}[\alpha]\,\epsilon^2 - (\lambda - 1)\,\text{sin}\,\alpha == 0$$

Solving each of these for **(- 1)**, we get

$$(\lambda - 1) == \frac{\text{Cos}[\alpha]^3\,\epsilon^2 + \text{Sin}[\alpha]^3\,\epsilon^2 + \eta 1\,\text{Sin}[\alpha]^2\,\epsilon^2 + \eta 2\,\text{Sin}[\alpha]^2\,\epsilon^2}{\text{Cos}[\alpha]}$$

and

$$(\lambda - 1) == \frac{1}{\text{Sin}[\alpha]}\,(\text{Cos}[\alpha]^3\,\epsilon^2 + \eta 2\,\text{Cos}[\alpha]^2\,\epsilon^2 + \text{Cos}[\alpha]\,\text{Sin}[\alpha]^2\,\epsilon^2 + \eta 1\,\text{Cos}[\alpha]\,\text{Sin}[\alpha]\,\epsilon^2)$$

Equating these we get

$$\frac{\text{Cos}[\alpha]^3\,\epsilon^2 + \text{Sin}[\alpha]^3\,\epsilon^2 + \eta 1\,\text{Sin}[\alpha]^2\,\epsilon^2 + \eta 2\,\text{Sin}[\alpha]^2\,\epsilon^2}{\text{Cos}[\alpha]} == \frac{1}{\text{Sin}[\alpha]}\,(\text{Cos}[\alpha]^3\,\epsilon^2 + \eta 2\,\text{Cos}[\alpha]^2\,\epsilon^2 + \text{Cos}[\alpha]\,\text{Sin}[\alpha]^2\,\epsilon^2 + \eta 1\,\text{Cos}[\alpha]\,\text{Sin}[\alpha]\,\epsilon^2)$$

We note that **this equation is the same as** Equation **(7)**, which means that it's leading term for small **, (1 = 0 , 2 = 0**), is

$$\text{t}[\alpha]$$

and therefore is satisfied for $\alpha = \alpha i$, **i = 1,2,3,4**.

So we may use **either** one of the above expressions for **(- 1)** . This makes sense since none of the **i** 's are equal to or $_2$.

We next substitute the 1st expression for **(- 1)** into the 3rd and 4th elements of Equation (8)**.**

$$\Big(\text{Cos}[\alpha]^3\,\epsilon^2 + \text{Sin}[\alpha]^3\,\epsilon^2 + \eta 2\,\text{Sin}[\alpha]^2\,\epsilon^2 + \eta 1\,\text{Cos}[\alpha]\,\text{Sin}[\alpha]\,\epsilon^2 + \eta 1 - \eta 1\,(\lambda - 1)\ /.\ (\lambda\ -\ 1) \to \frac{\text{Cos}[\alpha]^3\,\epsilon^2 + \text{Sin}[\alpha]^3\,\epsilon^2 + \eta 1\,\text{Sin}[\alpha]^2\,\epsilon^2 + \eta 2\,\text{Sin}[\alpha]^2\,\epsilon^2}{\text{Cos}[\alpha]}\Big) == 0$$

```
(Cos[α]^3 ϵ^2 + Sin[α]^3 ϵ^2 + η2 Sin[α]^2 ϵ^2 + η1 η2 Sin[α] ϵ^2 +
    η1 Cos[α] Sin[α] ϵ^2 + η2 - η2 (λ - 1) /.
       (λ - 1) ->
    (Cos[α]^3 ϵ^2 + Sin[α]^3 ϵ^2 + η1 Sin[α]^2 ϵ^2 + η2 Sin[α]^2 ϵ^2) / Cos[α]) == 0
```

It is not hard to see from these equations that **1** **0** and **2** **0** as **0**. Therefore the leading terms in their series will be

η1 - ϵ^2 **Cos**[α]^3 – ϵ^2 **Sin**[α]^3 and **η2** - ϵ^2 **Cos**[α]^3 – ϵ^2 **Sin**[α]^3 . Therefore we define the following.

```
η1[α_, ϵ_] = - ϵ^2 Cos[α]^3 - ϵ^2 Sin[α]^3;
```

```
η2[α_, ϵ_] = - ϵ^2 Cos[α]^3 - ϵ^2 Sin[α]^3;
```

From this we see that the leading term for **(- 1)** $\frac{\text{Cos}[\alpha]^3\, \epsilon^2 + \text{Sin}[\alpha]^3\, \epsilon^2}{\text{Cos}[\alpha]}$, where we have used the first of the two expressions for **(- 1)**. Therefore we define

```
λ[α_, ϵ_] = 1 + (Cos[α]^3 ϵ^2 + Sin[α]^3 ϵ^2) / Cos[α];
```

These three functions, **η1[α_,ϵ_]**, **η2[α_,ϵ_]**, and **λ[α_,ϵ_]**, will be used to start the iteration scheme for a continuation process which will construct the solution branches bifurcating from = 1. This will be done as follows.

```
eqs[x_, y_, u_, v_, λ_] :=
    {x^3 + x + y^3 + u y^2 + v y^2 - x λ == 0,
  x^3 + v x^2 + y^2 x + u y x + y - y λ == 0,
  x^3 + u y x + y^3 + v y^2 + 2 u - u λ == 0,
  x^3 + u y x + y^3 + v y^2 + 2 v + u v y - v λ == 0}
```

```
lhseqs[x_, y_, u_, v_, λ_] :=
    {x^3 + x + y^3 + u y^2 + v y^2 - x λ, x^3 + v x^2 + y^2 x + u y x + y - y λ,
  x^3 + u y x + y^3 + v y^2 + 2 u - u λ, x^3 + u y x + y^3 + v y^2 + 2 v + u v y - v λ}
```

We calculate the determinant of the Jacobian matrix of **lhseqs** so that we can see if it gets near zero. This would mean the map is becoming singular, and we may be near a bifurcation point.

```
jacobian[x_, y_, u_, v_, λ_] =
 {{∂x lhseqs[x, y, u, v, λ][[1]], ∂y lhseqs[x, y, u, v, λ][[1]],
   ∂u lhseqs[x, y, u, v, λ][[1]], ∂v lhseqs[x, y, u, v, λ][[1]]},
  {∂x lhseqs[x, y, u, v, λ][[2]], ∂y lhseqs[x, y, u, v, λ][[2]],
   ∂u lhseqs[x, y, u, v, λ][[2]], ∂v lhseqs[x, y, u, v, λ][[2]]},
  {∂x lhseqs[x, y, u, v, λ][[3]], ∂y lhseqs[x, y, u, v, λ][[3]],
   ∂u lhseqs[x, y, u, v, λ][[3]], ∂v lhseqs[x, y, u, v, λ][[3]]},
  {∂x lhseqs[x, y, u, v, λ][[4]], ∂y lhseqs[x, y, u, v, λ][[4]],
   ∂u lhseqs[x, y, u, v, λ][[4]], ∂v lhseqs[x, y, u, v, λ][[4]]}}
```

```
     3 x^2      1          3 y^2  2 u y  2 v y        y^2            y^2
3 x^2  2 v x  y^2  u y   u x  2 y x      1        x y            x^2
      3 x^2  u y          3 y^2  2 v y  u x    x y      2        y^2
      3 x^2  u y       3 y^2  2 v y  u v  u x  v y  x y  y^2  u y      2
```

```
jdet[{x_, y_, u_, v_}, λ_] = Det[jacobian[x, y, u, v, λ]];
```

```
findroot[{x0_, y0_, u0_, v0_}, λ_] :=
 {x, y, u, v} /. FindRoot[Evaluate[eqs[x, y, u, v, λ]],
   {x, x0}, {y, y0}, {u, u0}, {v, v0}, MaxIterations -> 20]
```

In order to follow a branch of solutions as we increment **λ**, we define the following function **bifbranch**. In the arguments of this function, **x0, y0, u0, v0, and λ00** represent the starting values of **x, y, u, v, and λ** respectively, with **dλ** being the size of the increment of **λ** and **n** the number of increments to perform .

```
bifbranch[x0_, y0_, u0_, v0_, λ00_, dλ_, n_] :=
 Module[{sol, jd},
 sol[0] := {x0, y0, u0, v0};
  sol[1] := findroot[sol[0], λ00 + dλ];
  sol[i_] :=
    (sol[i] = findroot[2 * sol[i - 1] - sol[i - 2], λ00 + i * dλ]) /;
     i >= 2;
  jd[i_] := jd[i] = jdet[sol[i], λ00 + i * dλ];
        Table[{λ00 + j * dλ, sol[j], jd[j]}, {j, 1, n}]
         ]
```

```
branch[α_, ϵ_, dλ_, n_] :=
    bifbranch[ϵ Cos[α], ϵ Sin[α],
  ϵ η1[α, ϵ], η2[α, ϵ], λ[α, ϵ], dλ, n]
```

We first find the branch for **α = α1.**

```
bbranch = branch[α1, .02, .0005, 650];
```

The following functions will enable us to plot each of the variables **x,y,u,v** versus λ.

```
plotsolx[α_, c_] :=
   ListPlot[Table[{bbranch[[i, 1]], bbranch[[i, 2, 1]]},
   {i, Length[bbranch]}
                        ],
      Axes -> Automatic,
  AxesLabel -> {λ, x}, PlotJoined -> True,
           PlotLabel -> StringForm["α=``", α],
  PlotRange -> All , PlotStyle -> Hue[c]]

plotsoly[α_, c_] :=
   ListPlot[Table[
  {bbranch[[i, 1]], bbranch[[i, 2, 2]]}, {i, Length[bbranch]}
                        ],
      Axes -> Automatic,
 AxesLabel -> {λ, y}, PlotJoined -> True,
           PlotLabel -> StringForm["α=``", α],
 PlotRange -> All , PlotStyle -> Hue[c]]

plotsolu[α_, c_] :=
   ListPlot[Table[
  {bbranch[[i, 1]], bbranch[[i, 2, 3]]}, {i, Length[bbranch]}
                        ],
      Axes -> Automatic,
 AxesLabel -> {λ, u}, PlotJoined -> True,
           PlotLabel -> StringForm["α=``", α],
 PlotRange -> All , PlotStyle -> Hue[c]]

plotsolv[α_, c_] :=
   ListPlot[Table[{bbranch[[i, 1]], bbranch[[i, 2, 4]]},
   {i, Length[bbranch]}
                        ],
      Axes -> Automatic,
  AxesLabel -> {λ, v}, PlotJoined -> True,
           PlotLabel -> StringForm["α=``", α],
  PlotRange -> All , PlotStyle -> Hue[c]]
```

```
plotsolx[α1, 0]
```

```
plotsoly[α1, 0]
```

```
plotsolu[α1, 0]
```

```
plotsolv[α1, 0]
```

We see from these graphs that the derivatives with respect to **λ** are getting large, which means that the map is becoming singular. In order to get around this difficulty we change variables. Instead of having **λ** as the independent variable we change to a new independent variable, say **x**. Now we can think of the unknowns being **(λ,y,u,v)** instead of **(x,y,u,v)**. We accomplish this with the following.

```
jacobianx[x_, y_, u_, v_, λ_] =
 {{∂λ lhseqs[x, y, u, v, λ][[1]], ∂y lhseqs[x, y, u, v, λ][[1]],
   ∂u lhseqs[x, y, u, v, λ][[1]], ∂v lhseqs[x, y, u, v, λ][[1]]},
  {∂λ lhseqs[x, y, u, v, λ][[2]], ∂y lhseqs[x, y, u, v, λ][[2]],
   ∂u lhseqs[x, y, u, v, λ][[2]], ∂v lhseqs[x, y, u, v, λ][[2]]},
  {∂λ lhseqs[x, y, u, v, λ][[3]], ∂y lhseqs[x, y, u, v, λ][[3]],
   ∂u lhseqs[x, y, u, v, λ][[3]], ∂v lhseqs[x, y, u, v, λ][[3]]},
  {∂λ lhseqs[x, y, u, v, λ][[4]], ∂y lhseqs[x, y, u, v, λ][[4]],
   ∂u lhseqs[x, y, u, v, λ][[4]], ∂v lhseqs[x, y, u, v, λ][[4]]}}
```

$$\begin{pmatrix} x & 3y^2 & 2uy & 2vy & y^2 & y^2 \\ y & ux & 2yx & 1 & xy & x^2 \\ u & 3y^2 & 2vy & ux & xy & 2 & y^2 \\ v & 3y^2 & 2vy & uv & ux & vy & xy & y^2 & uy & 2 \end{pmatrix}$$

```
jdetx[{λ_, y_, u_, v_}, x_] = Det[jacobianx[x, y, u, v, λ]];
```

```
findrootx[{λ00_, y0_, u0_, v0_}, x_] :=
 {λ, y, u, v} /. FindRoot[Evaluate[eqs[x, y, u, v, λ]],
   {λ, λ00}, {y, y0}, {u, u0}, {v, v0}, MaxIterations -> 20]
```

```
bifbranchx[λ00_, y0_, u0_, v0_, x0_, dx_, n_] :=
 Module[{sol, jd},
 sol[0] := {λ00, y0, u0, v0};
  sol[1] := findrootx[sol[0], x0 + dx];
  sol[i_] :=
    (sol[i] = findrootx[2 * sol[i - 1] - sol[i - 2], x0 + i * dx]) /;
     i >= 2;
  jd[i_] := jd[i] = jdetx[sol[i], x0 + i * dx];
        Table[{x0 + j * dx, sol[j], jd[j]}, {j, 1, n}]
         ]
```

We start this branch with the ending values in the previous calculation.

```
bbranch =
bifbranchx[1.3, 0.533956558414711235`,
   -0.228838525034876294`, -0.277231132406344116,
   0.533956558414711235`, .001, 3000];
```

We define a new set of plot functions for this case.

```
plotsolxx[α_, c_] :=
   ListPlot[Table[{bbranch[[i, 2, 1]], bbranch[[i, 1]]},
   {i, Length[bbranch]}
                        ],
      Axes -> Automatic,
  AxesLabel -> {λ, x}, PlotJoined -> True,
           PlotLabel -> StringForm["α=``", α],
  PlotRange -> All , PlotStyle -> Hue[c]]

plotsolxy[α_, c_] :=
   ListPlot[Table[{bbranch[[i, 2, 1]], bbranch[[i, 2, 2]]},
  {i, Length[bbranch]}
                        ],
      Axes -> Automatic,
 AxesLabel -> {λ, y}, PlotJoined -> True,
           PlotLabel -> StringForm["α=``", α],
 PlotRange -> All , PlotStyle -> Hue[c]]

plotsolxu[α_, c_] :=
   ListPlot[Table[{bbranch[[i, 2, 1]], bbranch[[i, 2, 3]]},
  {i, Length[bbranch]}
                        ],
      Axes -> Automatic,
 AxesLabel -> {λ, u}, PlotJoined -> True,
           PlotLabel -> StringForm["α=``", α],
 PlotRange -> All , PlotStyle -> Hue[c]]

plotsolxv[α_, c_] :=
   ListPlot[Table[{bbranch[[i, 2, 1]], bbranch[[i, 2, 4]]},
   {i, Length[bbranch]}
                        ],
      Axes -> Automatic,
  AxesLabel -> {λ, v}, PlotJoined -> True,
           PlotLabel -> StringForm["α=``", α],
  PlotRange -> All , PlotStyle -> Hue[c]]
```

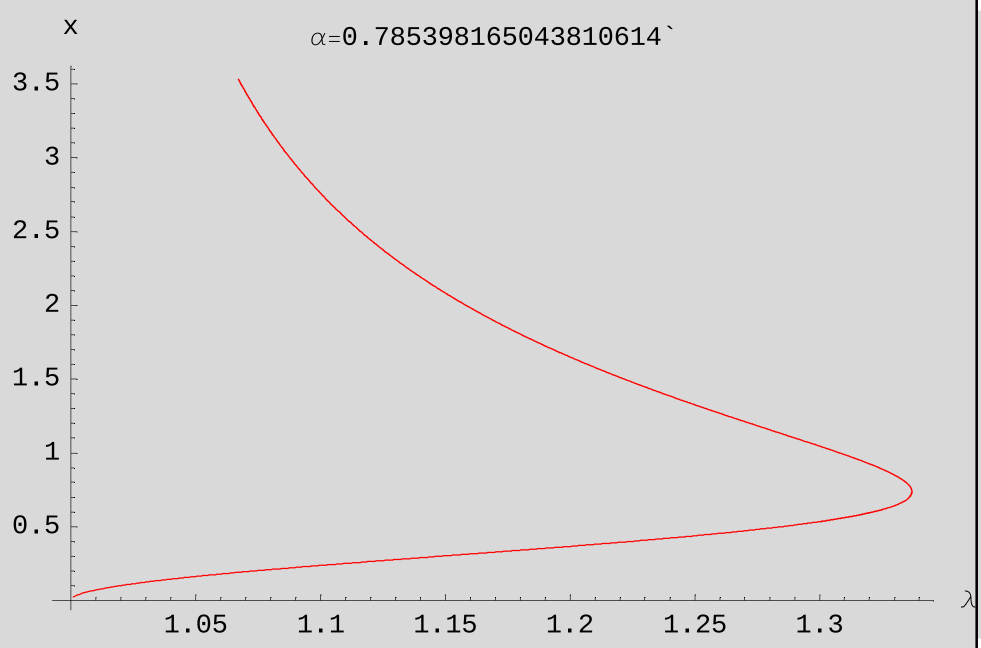
```
plotsolxx[α1, 0]
```

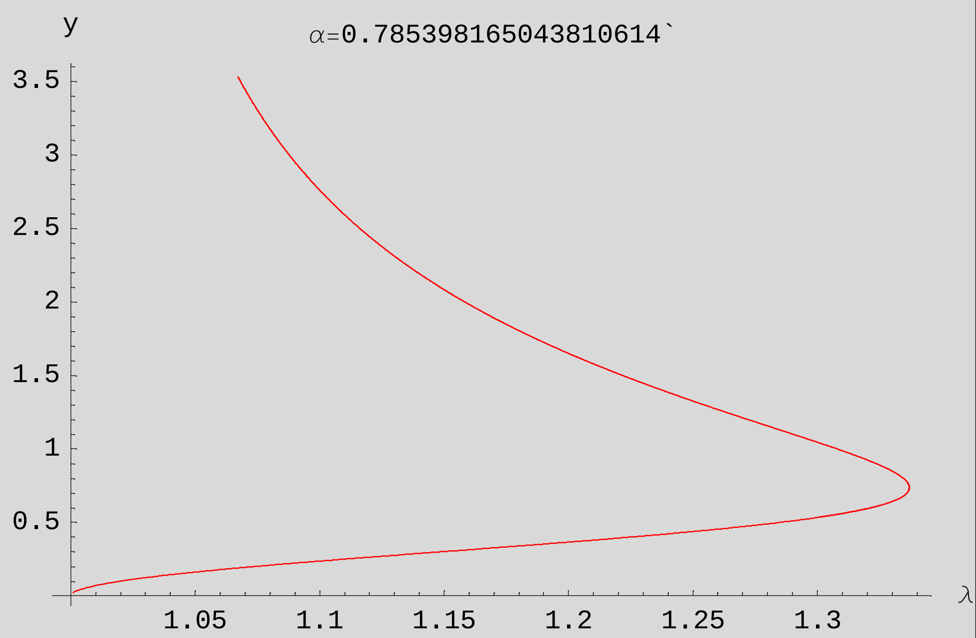
```
plotsolxy[α1, 0]
```

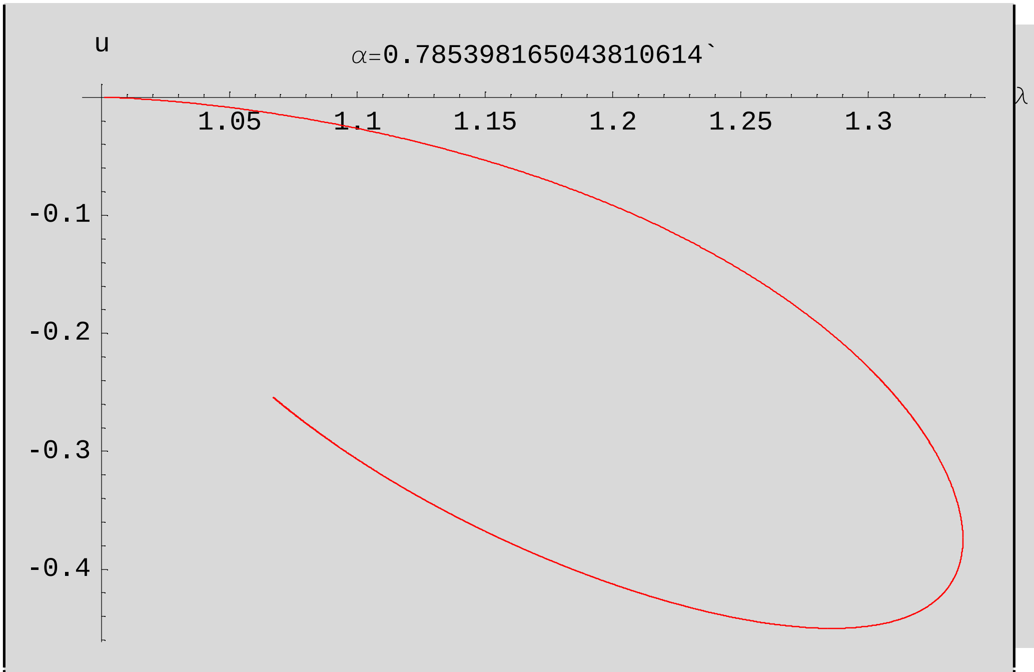
```
plotsolxu[α1, 0]
```

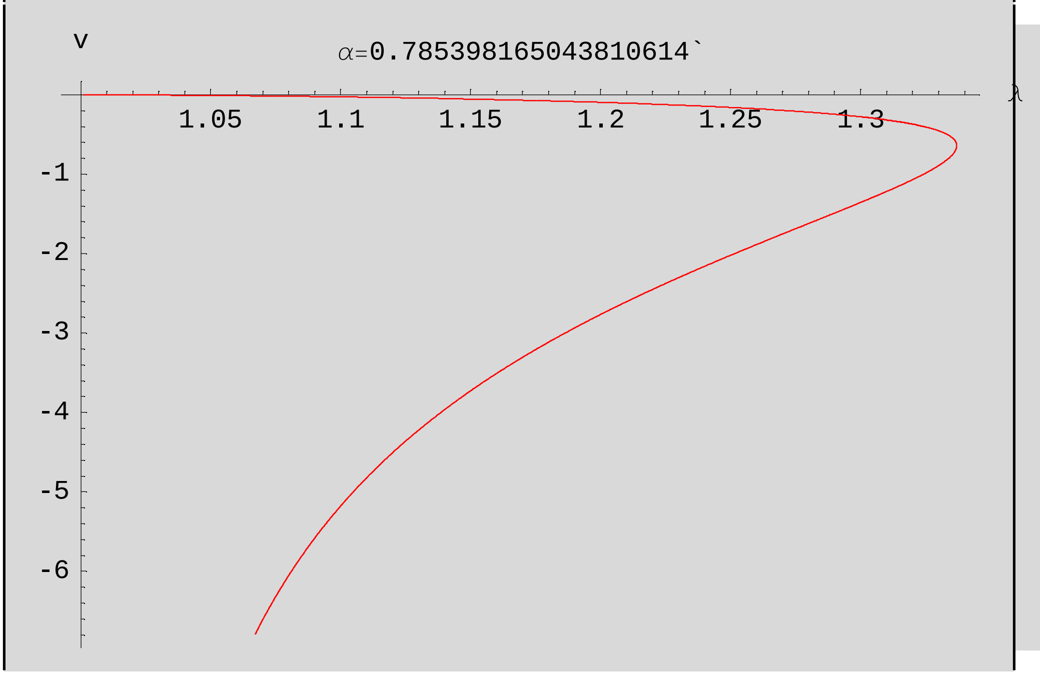
```
plotsolxv[α1, 0]
```

We join these graphs with the corresponding graphs from above.

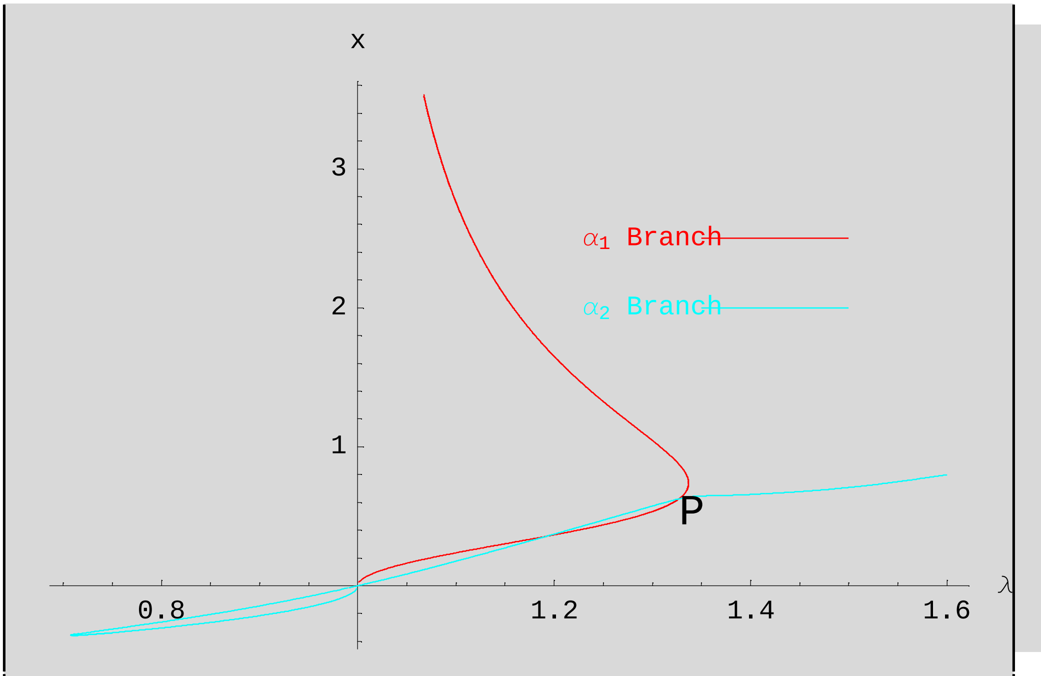


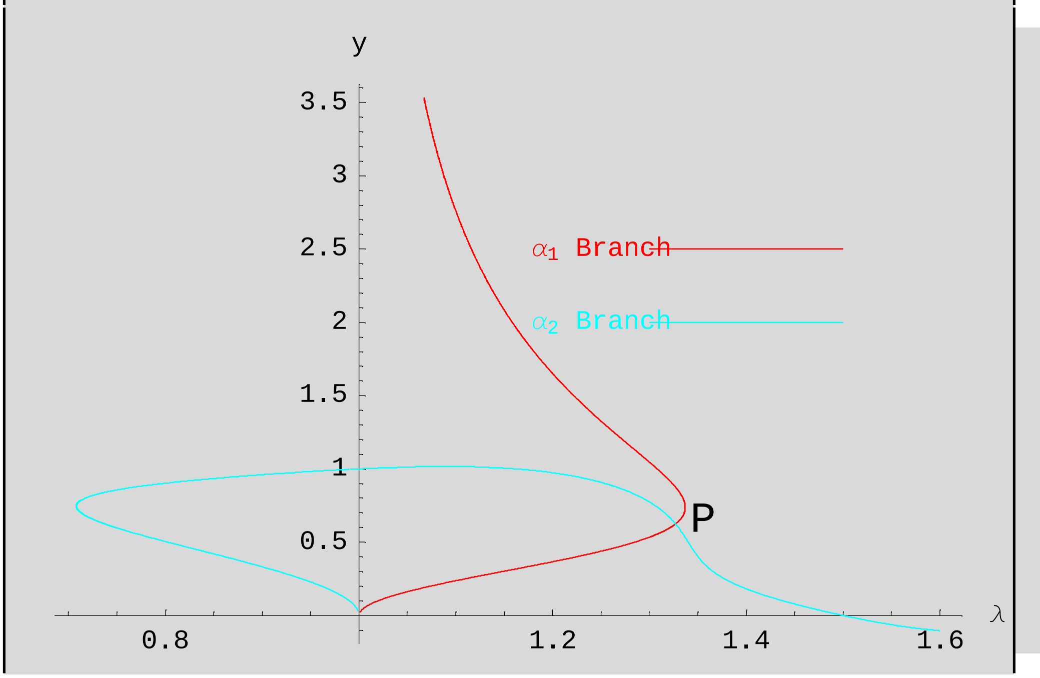


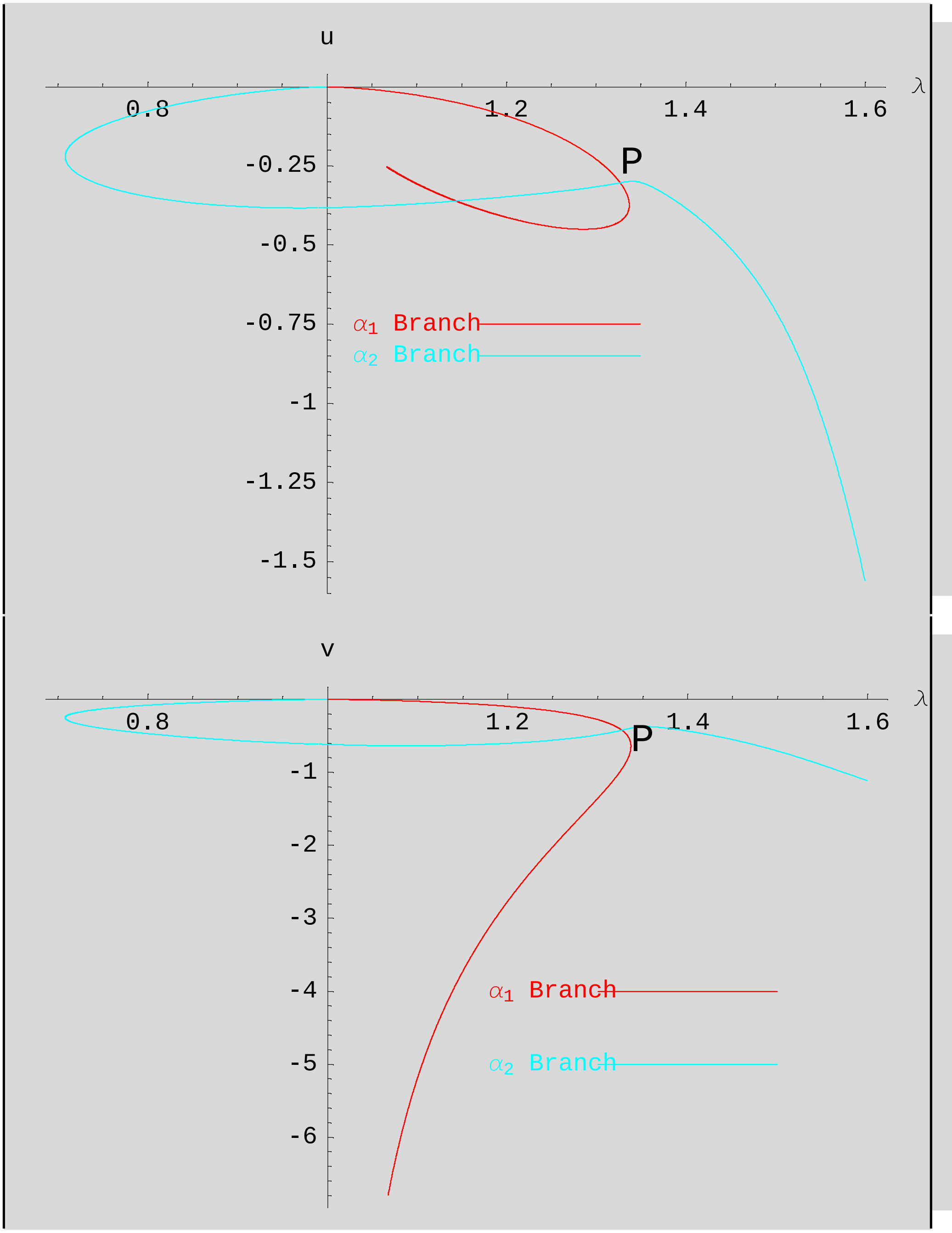




We perform a similar procedure for the branch corresponding to **α = α2.** Combining these results with the previous results for **α = α1,** we get
the following. **We note that the intersection point P is a bifurcation point.**


x
3
2
1
0.8
1.2
1.4
1.6
λ
α_1 Branch
α_2 Branch
P


y
3.5
3
2.5
2
1.5
1
0.5
0.8
1.2
1.4
1.6
λ
α_1 Branch
α_2 Branch
P



■ 2.2 An Example Where Bifurcation Does Not Occur

$$A\ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix};$$

$$w \quad \begin{pmatrix} x \\ y \\ u \\ v \end{pmatrix};$$

$$\mathbf{F}[\ ,w] \quad \begin{pmatrix} (x^2 \quad y^2 \quad u^2 \quad v^2)\,y \\ (x^2 \quad y^2 \quad u^2 \quad v^2)\,x \\ (x^2 \quad y^2 \quad u^2 \quad v^2)\,v \\ (x^2 \quad y^2 \quad u^2 \quad v^2)\,u \end{pmatrix};$$

```
A.w - λ w + F[λ, w] == 0                                        (*9*)
```

$$\begin{pmatrix} x \quad y\,(u^2 \quad v^2 \quad x^2 \quad y^2) \quad x \\ y \quad x\,(\ u^2 \quad v^2 \quad x^2 \quad y^2) \quad y \\ 2\,u \quad v\,(u^2 \quad v^2 \quad x^2 \quad y^2) \quad u \\ 2\,v \quad u\,(\ u^2 \quad v^2 \quad x^2 \quad y^2) \quad v \end{pmatrix} \quad 0$$

As in the previous example we will investigate whether bifurcation occurs from the double eigenvalue

```
λ0 = 1
```

$$w = \epsilon \begin{pmatrix} \mathrm{Cos}[\alpha] \\ \mathrm{Sin}[\alpha] \\ \eta 1 \\ \eta 2 \end{pmatrix}$$

Euation (3) then becomes

```
Dot[Flatten[{{Sin[α]}, {-Cos[α]}, {0}, {0}}], Flatten[{{(x^2 + y^2 + u^2 + v^2) y}, {-(x^2 + y^2 + u^2 + v^2) x}, {(x^2 + y^2 + u^2 + v^2) v}, {-(x^2 + y^2 + u^2 + v^2) u}} /.
    {x -> ϵ Cos[α], y -> ϵ Sin[α], u -> ϵ η1, v -> ϵ η2}]] == 0
```

$$-\epsilon\,\mathrm{Cos}[\alpha]^2\,(-\epsilon^2\,\eta 1^2 - \epsilon^2\,\eta 2^2 - \epsilon^2\,\mathrm{Cos}[\alpha]^2 - \epsilon^2\,\mathrm{Sin}[\alpha]^2) + \epsilon\,\mathrm{Sin}[\alpha]^2\,(\epsilon^2\,\eta 1^2 + \epsilon^2\,\eta 2^2 + \epsilon^2\,\mathrm{Cos}[\alpha]^2 + \epsilon^2\,\mathrm{Sin}[\alpha]^2) == 0$$

```
Simplify[%]
```

$$\epsilon^3\,(1 + \eta 1^2 + \eta 2^2) == 0$$

This shows that bifurcation does not occur since =0 is the only solution, which means only the zero solution, $\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$, of Equation (9) exists.

We note that in this example the nonlinearity is cubic as was the nonlinearity in the previous example where bifurcation does occur!

3. $L = \frac{d^2}{dx^2}$, $H = C^2[0,2\ ]$.

In this section we will be concerned with the **solutions on the interval [0 , 2]** of the differential equation

```
w'' + λw + f[t, w, w'] == 0
```

with the **periodic boundary conditions**

```
w[0] == w[2π]
```

```
w'[0] == w'[2π]
```

It is not hard to see that the linear problem

```
w'' + λw == 0
```

with the same boundary conditions, has the **eigenvalues** $\lambda = n^2$**, n = 0,1,2,3...**, with the **eigenfunctions 1 for n = 0 and {Cos[n t],Sin[n t]}, n = 1,2,3,... . We note that all the eigenvalues (except λ=0) have multiplicity 2.** In this example the nonlinearity will be

```
f[t_, y_, z_] := -k t y z^2 + y^2 z
```

Where **k** is a parameter. It will shown that when **k** < $\frac{4}{3}$ **no bifurcation will occur**, but when **k** $\frac{4}{3}$ **bifurcation does occur**. It is for this reason that we wish to investigate this problem. We want to learn what happens to the bifurcating branches of solutions as **k** decreases through the value $\frac{4}{3}$. **Do the solution branches just disappear or do they slowly collapse to zero?** We will learn what happens in the following. We investigate the bifurcation from the double eigenvalue

$$\lambda_0 = 1$$

Therefore dim **N** = **2** and dim $\mathbf{N}^{\perp}$ is **.** We will take a basis for **N** to be **{Cos[t],Sin[t]},** and represent the $\mathbf{N}^{\perp}$ component by η**[t]**.

We will use the inner product

```
⟨f_, g_⟩ := ∫_0^(2π) f[t] g[t] ⅆt
```

Therefore the vectors in the unit sphere(circle) in **N** can then be parametrized using the angle **α in [0 , 2)** by

```
y[α_][t_] := 1/√π (Cos[α] Sin[t] + Sin[α] Cos[t])
```

i.e.,

```
⟨y[α], y[α]⟩
```

It is easy to see that

```
yp[α_][t_] := 1/√π (Cos[α] Cos[t] - Sin[α] Sin[t])
```

is orthogonal to **y[α_][t_]** and so is in the tangent plane(line) to the unit sphere(circle) in **N** at **y[α_][t_]**.

i.e.,

```
⟨yp[α], y[α]⟩
```

Equation (1) then becomes

```
w[α_][t_] := ϵ (y[α][t] + η[t])
```

We need to define

```
f[α_][t_] =
 Simplify[f[t, y, z] /. {y -> w[α][t], z -> w[α]'[t]}]
```

then Equation (3) becomes

```
⟨yp[α], f[α]⟩ == 0
```

We obtain Equation (6) by letting **η[t]->0 and η′[t]->0** in the previous equation.

```
Simplify[% /. {η[t] -> 0, η′[t] -> 0}]
```

We wish to calculate the zeros of this equation, so let,

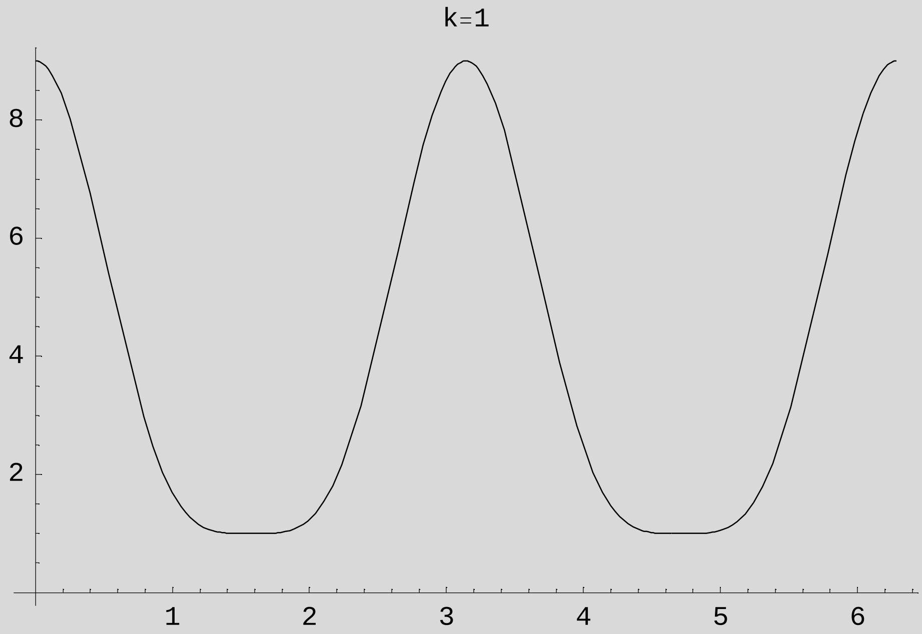
```
t[k_, α_] = 4 + 4 k Cos[2 α] + k Cos[4 α]
```

In order to see how the zeros of this function behave as a **k** varies we do the following.

```
plt[k_] :=
    Plot[t[k, α], {α, 0, 2π},
  PlotLabel -> StringForm["k=``", k],
    AxesOrigin -> {0, 0}]

tableplt[kmax_] := Table[plt[k], {k, 1, kmax, .05}]
```

```
tableplt[1.5]
```




```
{ Graphics , Graphics , Graphics ,
  Graphics , Graphics , Graphics , Graphics ,
  Graphics , Graphics , Graphics , Graphics }
```

We can animate these graphs and see that **t[k_,α_]** has no zeros for small values of **k (no bifurcation occurs)** and as **k** increases we get four zeros (**four bifurcating branches of solutions)**.

We can get an analytic representation of these zeros by the following.

```
Solve[t[k, α] == 0, α]
```

We define the real zeros of **t[k,α]**, as

$$\alpha 1[k\_] := \text{ArcCos}\left[\frac{(-4+3\,k)^{1/4}}{2^{3/4}\,k^{1/4}}\right]$$

$$\alpha 2[k\_] := \text{ArcCos}\left[-\frac{(-4+3\,k)^{1/4}}{2^{3/4}\,k^{1/4}}\right]$$

```
α3[k_] := α1[k] + π
```

```
α4[k_] := α2[k] + π
```

We note that for **k** = $\frac{4}{3}$ real zeros appear at $\frac{}{2}$ and $\frac{3}{2}$ **,** and for **k** > $\frac{4}{3}$ there are four real zeros. This agrees with the above graphs.

We note (as was mentioned in Section 2),that **α1[k]-α3[k]=** π and **α2[k]-α4[k] =** π, which means that the solution branches bifurcate in pairs in opposite directions. This is expected since the nonlinearity is cubic, so that if **w[t]** is a solution then **-w[t]** is also a solution.

■ 3.1 Constructing the Solution Branches

We will be interested in the bifurcation from the eigenvalue

```
λ0 = 1;
```

We first find η**[t]** for small . Equation (4) can be written as

```
η''[t] + λ0 η[t] == (λ0 − λ) (y[α][t] + η[t]) −
  ϵ² f[t, y[α][t] + η[t], y[α]'[t] + η'[t]]
```

Using Lemma 3.5 in [**1**], which says that η**->0 as** ϵ**->0**, it is not hard to see that the leading term in the approximation for η**[t]** satisfies the following approximation to the above equation.

```
η''[t] + λ0 η[t] == (λ0 − λ) y[α][t] − ϵ² f[t, y[α][t], y[α]'[t]]
```

Next let

```
ff[α_][t_] = Simplify[f[t, y[α][t], y[α]'[t]]]
```

In order for a solution η**[t]** of the previous differential equation to exist, the right-hand side must be orthogonal to both of the eigenfunctions **Sin[t]** and **Cos[t]**, i.e.,

```
(λ0 − λ) ⟨y[α], Sin⟩ − ϵ² ⟨ ff[α], Sin⟩ == 0
```

and,

```
(λ0 − λ) ⟨y[α], Cos⟩ − ϵ² ⟨ ff[α], Cos⟩ == 0
```

Solving each of these two equations for **(1 -)** we get

$$(1-\lambda) = \frac{\epsilon^2 \left(-\frac{1}{4} k \sqrt{\pi}\, \mathrm{Cos}[\alpha] - \frac{\mathrm{Sin}[\alpha]}{4\sqrt{\pi}} + \frac{k\,\mathrm{Sin}[\alpha]}{8\sqrt{\pi}} - \frac{k\,\mathrm{Sin}[3\alpha]}{16\sqrt{\pi}}\right)}{\sqrt{\pi}\,\mathrm{Cos}[\alpha]}$$

and

$$(1-\lambda) = \frac{\epsilon^2 \left(\frac{\mathrm{Cos}[\alpha]}{4\sqrt{\pi}} + \frac{k\,\mathrm{Cos}[\alpha]}{8\sqrt{\pi}} + \frac{3k\,\mathrm{Cos}[3\alpha]}{16\sqrt{\pi}} - \frac{1}{4} k \sqrt{\pi}\,\mathrm{Sin}[\alpha]\right)}{\sqrt{\pi}\,\mathrm{Sin}[\alpha]}$$

Equating these two equations we get

$$\mathrm{Simplify}\left[\frac{\epsilon^2 \left(-\frac{1}{4} k \sqrt{\pi}\, \mathrm{Cos}[\alpha] - \frac{\mathrm{Sin}[\alpha]}{4\sqrt{\pi}} + \frac{k\,\mathrm{Sin}[\alpha]}{8\sqrt{\pi}} - \frac{k\,\mathrm{Sin}[3\alpha]}{16\sqrt{\pi}}\right)}{\sqrt{\pi}\,\mathrm{Cos}[\alpha]} == \frac{\epsilon^2 \left(\frac{\mathrm{Cos}[\alpha]}{4\sqrt{\pi}} + \frac{k\,\mathrm{Cos}[\alpha]}{8\sqrt{\pi}} + \frac{3k\,\mathrm{Cos}[3\alpha]}{16\sqrt{\pi}} - \frac{1}{4} k \sqrt{\pi}\,\mathrm{Sin}[\alpha]\right)}{\sqrt{\pi}\,\mathrm{Sin}[\alpha]}\right]$$

Since **Csc[α] Sec[α] ≠ 0**, we can write this result as

```
4 + 4 k Cos[2 α] + k Cos[4 α] == 0
```

The left-hand side is exactly what we had defined previously as **t[k_, α_]** and therefore is satisfied for **α=αi, i=1,2,3,4**. So that we may use either one of the above expressions for **(1-λ)**, and we can now solve the above differential equation for **η[t]** as follows.

```
dsol = DSolve[η''[t] + λ0 η[t] == Simplify[
      (λ0 - λ) y[α][t] - ϵ^2 f[t, y[α][t], y[α]'[t]]], η[t], t];
```

```
η[t_, α_, λ_, ϵ_, k_] = η[t] /. Flatten[dsol]
```

We next solve for **λ** in the first of the two expression above for **(1-λ)**, and define,

$$\lambda[\alpha\_, \epsilon\_, k\_] = -\frac{\epsilon^2 \left(-\frac{1}{4} k \sqrt{\pi}\, \mathrm{Cos}[\alpha] - \frac{\mathrm{Sin}[\alpha]}{4\sqrt{\pi}} + \frac{k\,\mathrm{Sin}[\alpha]}{8\sqrt{\pi}} - \frac{k\,\mathrm{Sin}[3\alpha]}{16\sqrt{\pi}}\right)}{\sqrt{\pi}\,\mathrm{Cos}[\alpha]} + 1;$$

Substituting this into **η[t_,α_,λ_,ϵ_,k_]**, we get

```
η[t_, α_, ϵ_, k_] = Simplify[η[t, α, λ[α, ϵ, k], ϵ, k]]
```

We next show that **η[t_,α_,ϵ_,k_]** satisfies our periodic boundary conditions, i.e.,

```
η[0, α, ϵ, k] - η[2 π, α, ϵ, k] == 0
```

```
True
```

and,

```
(∂t η[t, α, ϵ, k] /. t -> 0) - (∂t η[t, α, ϵ, k] /. t -> 2 π) == 0
```

```
Simplify[%]
```

$$\frac{\epsilon^2\,(4+4\,k\,\mathrm{Cos}[2\,\alpha]+k\,\mathrm{Cos}[4\,\alpha])\,\mathrm{Sec}[\alpha]}{16\,\sqrt{\pi}} == 0$$

We see that this equation is satisfied since the expression in parenthesis is **t[k_, α_]** which is zero for **α = αi, i=1,2,3,4**.

Since η is in $\mathbf{N}^{\perp}$, we must impose the conditions that **η[t_,α_,ϵ_,k_]** is **orthogonal** to **Cos[t] and Sin[t]**. This is done by solving for **C[1]** and **C[2]** in the following. We first define,

```
ηη[t_] = η[t, α, ϵ, k];
```

so that we can impose the orthogonality conditions,

```
⟨ηη, Sin⟩ == 0
```

```
c[1] = C[1] /. Solve[%, C[1]]
```

and

```
⟨ηη, Cos⟩ == 0
```

```
c[2] = C[2] /. Solve[%, C[2]]
```

We next substitute these expressions for c[1] and c[2] into η[t_,α_,ϵ_,k_] to finally get our approximation to η and $\partial_t \eta$ for small , i.e.,

```
η[t_, α_, ϵ_, k_] =
 Simplify[η[t, α, ϵ, k] /. {C[1] -> c[1], C[2] -> c[2]}]
```

```
η1[t_, α_, ϵ_, k_] = ∂t η[t, α, ϵ, k]
```

The functions **η[t_,α_,ϵ_,k_], η1[t_,α_,ϵ_,k_], λ[α_,ϵ_,k_], α1[k_], and α2[k_]** will now be used to start the iteration scheme for a continuation process which will construct the solution branches bifurcating from λ_0 **= 1**. This will be analogous to the procedure used in Section 2 for algebraic equations, except that we must now use **Newton's Method** instead of the built-in function **FindRoot**.We first substitute the expressions for **α1[k_] and α2[k_],** that we derived previously, i.e.

```
α1[k_] := ArcCos[(-4 + 3 k)^(1/4) / (2^(3/4) k^(1/4))]
α2[k_] := ArcCos[-(-4 + 3 k)^(1/4) / (2^(3/4) k^(1/4))]
```

into the functions **η[t_,α_,ϵ_,k_], η1[t_,α_,ϵ_,k_],** and **λ[α_,ϵ_,k_]** and then simplify.

```
ηα1[t_, ϵ_, k_] = Simplify[η[t, α1[k], ϵ, k] //.
   {Sin[a_ + b_] -> Sin[a] Cos[b] + Sin[b] Cos[a],
    Cos[a_ + b_] -> Cos[a] Cos[b] - Sin[a] Sin[b], Sin[2 a_] ->
     2 Sin[a] Cos[a], Cos[2 a_] -> Cos[a]^2 - Sin[a]^2,
    Sin[4 a_] -> 4 Cos[a] Sin[a] (Cos[a]^2 - Sin[a]^2),
    Cos[4 a_] -> -1 + 2 (Cos[a]^2 - Sin[a]^2)^2,
    Sin[2 ArcCos[a_]] -> 2 a Sqrt[1 - a^2],
    Sin[4 ArcCos[a_]] -> 4 a Sqrt[1 - a^2] (2 a^2 - 1)}]
```

```
η1α1[t_, ϵ_, k_] = Simplify[η1[t, α1[k], ϵ, k] //.
   {Sin[a_ + b_] -> Sin[a] Cos[b] + Sin[b] Cos[a],
    Cos[a_ + b_] -> Cos[a] Cos[b] - Sin[a] Sin[b], Sin[2 a_] ->
     2 Sin[a] Cos[a], Cos[2 a_] -> Cos[a]^2 - Sin[a]^2,
    Sin[4 a_] -> 4 Cos[a] Sin[a] (Cos[a]^2 - Sin[a]^2),
    Cos[4 a_] -> -1 + 2 (Cos[a]^2 - Sin[a]^2)^2,
    Sin[2 ArcCos[a_]] -> 2 a Sqrt[1 - a^2],
    Sin[4 ArcCos[a_]] -> 4 a Sqrt[1 - a^2] (2 a^2 - 1)}]
```

```
λα1[ϵ_, k_] = FullSimplify[λ[α1[k], ϵ, k] /.
   Sin[3 a_] -> 2 Cos[a]^2 Sin[a] + Sin[a] (Cos[a]^2 - Sin[a]^2)]
```

```
ηα2[t_, ϵ_, k_] = Simplify[η[t, α2[k], ϵ, k] //.
   {Sin[a_ + b_] -> Sin[a] Cos[b] + Sin[b] Cos[a],
    Cos[a_ + b_] -> Cos[a] Cos[b] - Sin[a] Sin[b], Sin[2 a_] ->
     2 Sin[a] Cos[a], Cos[2 a_] -> Cos[a]^2 - Sin[a]^2,
    Sin[4 a_] -> 4 Cos[a] Sin[a] (Cos[a]^2 - Sin[a]^2),
    Cos[4 a_] -> -1 + 2 (Cos[a]^2 - Sin[a]^2)^2,
    Sin[2 ArcCos[a_]] -> 2 a Sqrt[1 - a^2],
    Sin[4 ArcCos[a_]] -> 4 a Sqrt[1 - a^2] (2 a^2 - 1)}]
```

```
η1α2[t_, ϵ_, k_] = Simplify[η1[t, α2[k], ϵ, k] //.
   {Sin[a_ + b_] -> Sin[a] Cos[b] + Sin[b] Cos[a],
    Cos[a_ + b_] -> Cos[a] Cos[b] - Sin[a] Sin[b], Sin[2 a_] ->
     2 Sin[a] Cos[a], Cos[2 a_] -> Cos[a]^2 - Sin[a]^2,
    Sin[4 a_] -> 4 Cos[a] Sin[a] (Cos[a]^2 - Sin[a]^2),
    Cos[4 a_] -> -1 + 2 (Cos[a]^2 - Sin[a]^2)^2,
    Sin[2 ArcCos[a_]] -> 2 a Sqrt[1 - a^2],
    Sin[4 ArcCos[a_]] -> 4 a Sqrt[1 - a^2] (2 a^2 - 1)}]
```

```
λα2[ϵ_, k_] = FullSimplify[λ[α2[k], ϵ, k] /.
   Sin[3 a_] -> 2 Cos[a]^2 Sin[a] + Sin[a] (Cos[a]^2 - Sin[a]^2)]
```

We next calculate the expressions for **ηα1, η1α1, ηα2,** and **η1α2** at **t=0**.

```
ηα10[ϵ_, k_] = Simplify[ηα1[0, ϵ, k]]
```

```
η1α10[ϵ_, k_] = Simplify[η1α1[0, ϵ, k]]
```

```
ηα20[ϵ_, k_] = Simplify[ηα2[0, ϵ, k]]
```

```
η1α20[ϵ_, k_] = Simplify[η1α2[0, ϵ, k]]
```

Next consider the following initial-value problem that is associated with our boundary-value problem. We will use **Newton's Method** to find the values of **a** and **b** that will make the solutions of this **initial-value problem** satisfy our **boundary-value problem**. In order to accomplish this we will need the derivatives with respect to **a** and **b** of the solution of this initial-value problem. This will lead to a system of three second order ordinary equations together with six initial conditions, as follows.

```
eqs =
{w''[t, a, b] + λ w[t, a, b] +
    f[t, w[t, a, b], w'[t, a, b]] == 0,
                w[0, a, b] == a, w'[0, a, b] == b}
```

The solution **w[t, a, b]** of this initial-value problem will satisfy our boundary conditions if **a** and **b** satisfy the equations

```
a - w[2 π, a, b] == 0
b - w'[2 π, a, b] == 0
```

The following will make the notation easier.

```
cat[x_, y_] :=
 ToExpression[StringJoin[ToString[x], ToString[y]]]

newnames = {∂a q_''[t, a, b] :→ cat[q, d][1]''[t],
                  ∂b q_''[t_, a, b] :→ cat[q, d][2]''[t],
                  ∂a q_'[t_, a, b] :→ cat[q, d][1]'[t],
                  ∂b q_'[t_, a, b] :→ cat[q, d][2]'[t],
                  ∂a q_[t_, a, b] :→ cat[q, d][1][t],
                  ∂b q_[t_, a, b] :→ cat[q, d][2][t],
                     q_[t_, a, b] :→ q[t]};
```

```
diffeqs[a_, b_, λ_, k_] =
Flatten[{eqs, ∂a eqs, ∂b eqs} /. newnames]
```

The initial-value problem is solved with,

```
ivp[a_, b_, λ_, k_] :=
  NDSolve[diffeqs[a, b, λ, k], {w, wd[1], wd[2]},
                  {t, 2 π} , MaxSteps -> 50000 ,
  AccuracyGoal -> 25, PrecisionGoal -> 12]
```

Newton's Method will be formulated as follows.

```
jacobian =
    {{1 - wd[1][2 π], -wd[2][2 π]},
        {-wd[1]'[2 π], 1 - wd[2]'[2 π]}};
```

```
y1[a_, b_] :=
    {{a - w[2 π]},
        {b - w'[2 π]}}
```

```
dy1[a_, b_] := -Inverse[jacobian].y1[a, b];
```

```
ff[{a_, b_}, λ_, k_] :=
                    (ivp1 = ivp[a, b, λ, k];
                     {a, b} + Flatten[dy1[a, b] /. ivp1]);
```

The iteration scheme for Newton's Method then becomes the following function, with **l** keeping track of how many iterates it takes to get within an error of ε to a fixed-point. It is important to know how many iterates it takes, since if this number starts to increase it may mean that we are approaching a bifurcation point on the branch we are following.

```
fp[{a_, b_}, λ_ , k_, ε_] := (l = 0;
                FixedPoint[(l++; ff[#, λ, k]) &, {a, b},
                 SameTest -> ( (Abs[ #1[[1]] - #2[[1]] ] +
                        Abs[ #1[[2]] - #2[[2]] ]) < ε &)
                           ]
                            )
```

We will need the determinate of the above Jacobian matrix in order to check if it gets close to zero. This would mean that the mapping is becoming singular and that we may be approaching a bifurcation point on the branch that we are following (see previous remark concerning the number of iterates).

```
jd[ivp1_] :=
             Det[jacobian /. Flatten[ivp1] ];
```

In order to follow a branch of solutions as we increment **λ**, we define the following function **idbranch**. In the arguments of this function, **a0**, **b0**, and **λ0** represent the starting values of **a**, **b**, and **λ** respectively, with **dλ** being the size of the increment of **λ** and **jm** the number of increments to perform .

```
idbranch[a0_, b0_, λ0_, dλ_, jm_, ε_, k_] :=
  Module[{idb, jbd},
    idb[0] := {a0, b0};
    idb[1] := fp[idb[0], λ0, k, ε];
    idb[i_] :=
               (idb[i] = fp[(2 idb[i - 1] - idb[i - 2]),
      λ0 + (i - 1) dλ, k, ε]) /; i >= 2 ;
    jbd[i_] := jbd[i] = jd[ivp1];
     Table[{λ0 + (j - 1) dλ, idb[j], jbd[j], l}, {j, jm}] ]
```

In order to start this continuation process we now use the previously defined functions, **y[α][t], ηα10[ϵ,k], η1α10[ϵ,k], ηα20[ϵ,k], η1α20[ϵ,k], λα1, λα2, α1**, and **α2**. We recall that in this notation the **α1** and **α2**, in the names**,**
refer to the branches bifurcating in the direction of the angles **α1** and **α2** respectively. The starting values at **t = 0** are defined as follows.

```
w[t_, a_, α_, ϵ_] := ϵ (y[α][t] + a)
w'[t_, b_, α_, ϵ_] := ϵ (y[α]'[t] + b)
```

```
wα1[ϵ_, k_] = w[0, ηα10[ϵ, k], α1[k], ϵ];
w1α1[ϵ_, k_] = w'[0, η1α10[ϵ, k], α1[k], ϵ];
wα2[ϵ_, k_] = w[0, ηα20[ϵ, k], α2[k], ϵ];
w1α2[ϵ_, k_] = w'[0, η1α20[ϵ, k], α2[k], ϵ];
```

We use these to define functions **branch1** and **branch2** which will allow us to construct the branches of solutions bifurcating in the direction of the angles **α1** and **α2** respectively.

```
branch1[ϵ_, k_, jm_] :=
idbranch[wα1[ϵ, k],
  w1α1[ϵ, k], λα1[ϵ, k], .005, jm, 10^-8, k]
```

```
branch2[ϵ_, k_, jm_] :=
idbranch[wα2[ϵ, k],
  w1α2[ϵ, k], λα2[ϵ, k], .005, jm, 10^-8, k]
```

For example we can construct the branches starting at **ϵ = .15 for k = 1.5** with **450** steps as follows.

```
bran1 = branch1[.15, 1.5, 450];
```

```
bran2 = branch2[.15, 1.5, 450];
```

We would like to use **ParametricPlot3D** in order to represent these results graphically, but to do this we first have to convert the values of **a** and **b** into functions of **λ**. This is accomplished as follows.

```
alam1 = Interpolation[Table[
   {bran1[[i, 1]], bran1[[i, 2, 1]]}, {i, Length[bran1]}]]
```

```
blam1 = Interpolation[Table[
   {bran1[[i, 1]], bran1[[i, 2, 2]]}, {i, Length[bran1]}]]
```

```
alam2 = Interpolation[Table[
   {bran2[[i, 1]], bran2[[i, 2, 1]]}, {i, Length[bran2]}]]
```

```
blam2 = Interpolation[Table[
   {bran2[[i, 1]], bran2[[i, 2, 2]]}, {i, Length[bran2]}]]
```

```
branches =
    ParametricPlot3D[{{λ, alam1[λ], blam1[λ], Hue[0]},
   {λ, alam2[λ], blam2[λ], Hue[.5]}},
  {λ, 1.00907807987241016`, 3.25279692012759014`},
  AxesLabel -> {λ, a, b}]
```

We next wish to investigate what happens as, **k -> 4/3,** and the corresponding branches coalesce, since **α1->α2**.

```
wklim[ϵ_] = Limit[wα1[ϵ, k], k -> 4 / 3]
```

We also see, as expected, that **wα2** has the same limit.

```
Limit[wα2[ϵ, k], k -> 4 / 3]
```

```
w1klim[ϵ_] = Limit[w1α1[ϵ, k], k -> 4 / 3]
```

And also **w1α2** correspondingly has this same limit.

```
Limit[w1α2[ϵ, k], k -> 4 / 3]
```

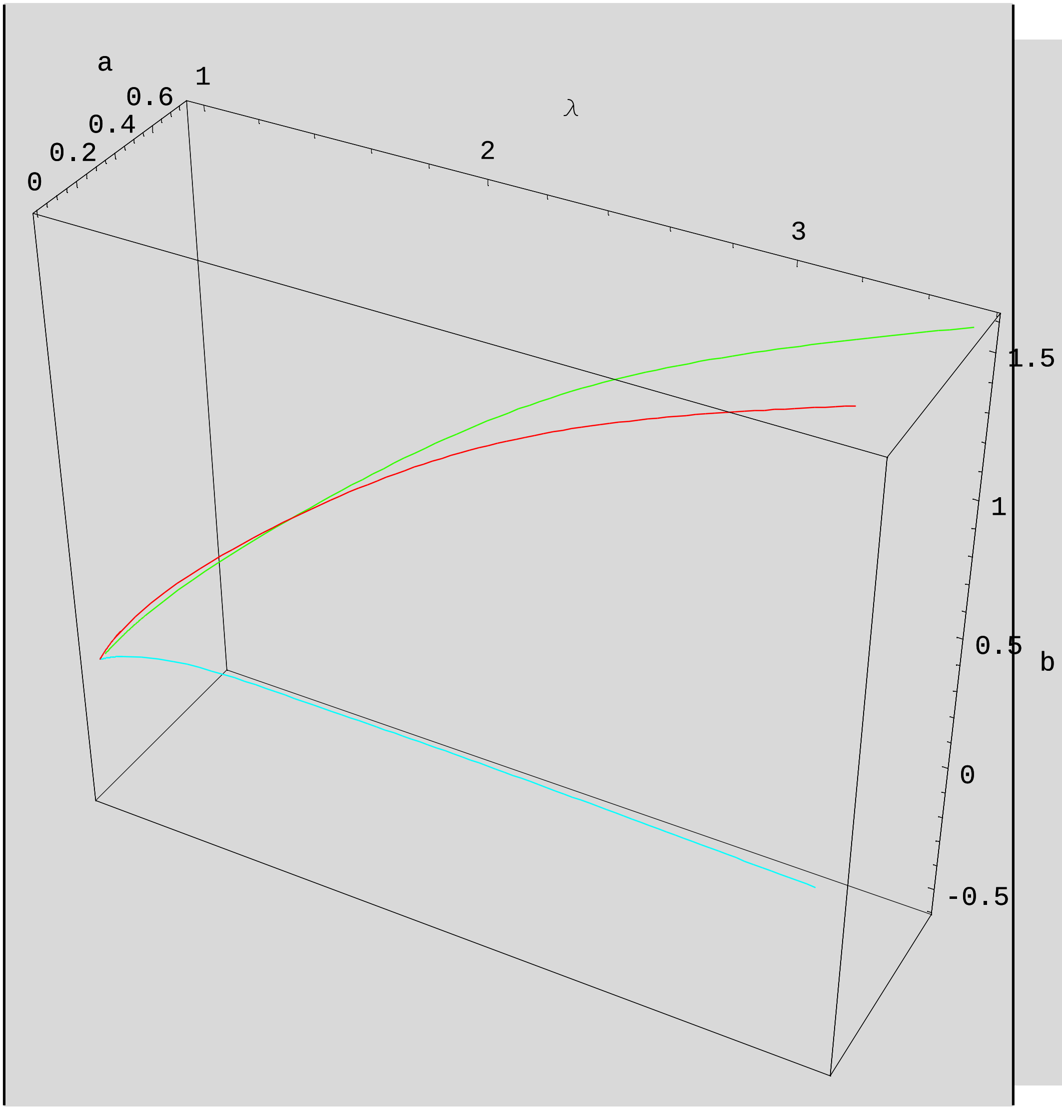
```
λklim[ϵ_] = Limit[λα1[ϵ, k], k -> 4 / 3]
```

We can use these as starting values to calculate the limit branch.

```
branchklim[ϵ_] :=
idbranch[wklim[ϵ], w1klim[ϵ],
  λklim[ϵ], .0005, 5000, 10^-8, N[4 / 3]]
```

```
branklim = branchklim[.05];
```

We can plot this limiting branch, in **green (Hue[.3])**, together with the previous two branches corresponding to **k = 1.5** as follows.



In summary we see that as `k -> 4/3` the two branches will come together and then there will be no bifurcation from `λ = 1` for `k < 4/3`. This corresponds to the real zeros of `t[k_,α_]` disappearing for `k < 4/3` as mentioned previously.

■ 3.2 Animating the Solutions Corresponding to Solution Branches

We now illustrate the previous results by using animation. This is done by using the data for **a** and **b** in the two solution branches **bran1** and **bran2**. These values are used as the initial data in solving the original differential equation as follows.

```
diffeqs[a_, b_, λ_, k_] =
{w''[t] + λ w[t] + f[t, w[t], w'[t]] == 0,
                    w[0] == a, w'[0] == b}
```

```
ivp2[a_, b_, λ_, k_] :=
  NDSolve[diffeqs[a, b, λ, k], w,
                    {t, 2 Pi} , MaxSteps -> 50000 ,
   AccuracyGoal -> 25, PrecisionGoal -> 12];
```

```
solution[a_, b_, λ_, k_, t_] :=
 w[t] /. Flatten[ivp2[a, b, λ, k]]
```

The solutions are plotted as follows.

```
plotsolutions := Plot[Evaluate[
             {solution[
                     bran1[[i, 2, 1]],
     bran1[[i, 2, 2]], bran1[[i, 1]], 1.5, t],
                 solution[
                     bran2[[i, 2, 1]],
     bran2[[i, 2, 2]], bran2[[i, 1]], 1.5, t]}
                                     ],
        {t, 0, 2 π}, Axes -> Automatic,
  AxesLabel -> {t, W}, PlotRange -> {-1.5, 1.6},
        PlotStyle -> {Hue[0], Hue[.5]},
 PlotLabel -> StringForm["λ=``", bran1[[i, 1]]],
  DisplayFunction -> Identity]
```

Next we designate a point on each of the branches for **`k = 1.5`**. These were plotted previously with the name "**`branches`**". These points correspond to the **ith** element in the lists of data **`bran1`** and **`bran2`**.

```
plotbranches := Show[{branches, Graphics3D[
{{PointSize[.03],
                        Point[
       {bran1[[i, 1]], bran1[[i, 2, 1]], bran1[[i, 2, 2]]}]},
    {PointSize[.03],
                        Point[
                            {bran2[[i, 1]],
        bran2[[i, 2, 1]], bran2[[i, 2, 2]]}]}
}                                             ]
                              },
  DisplayFunction -> Identity]
```

The following command creates a table of **`GraphicsArray`**'s which can be animated easily.

```
animatesolnbranches[imin_, imax_, di_] :=
Table[
  Show[GraphicsArray[{{plotsolutions}, {plotbranches}}]],
  {i, imin, imax, di}]
```

For example,

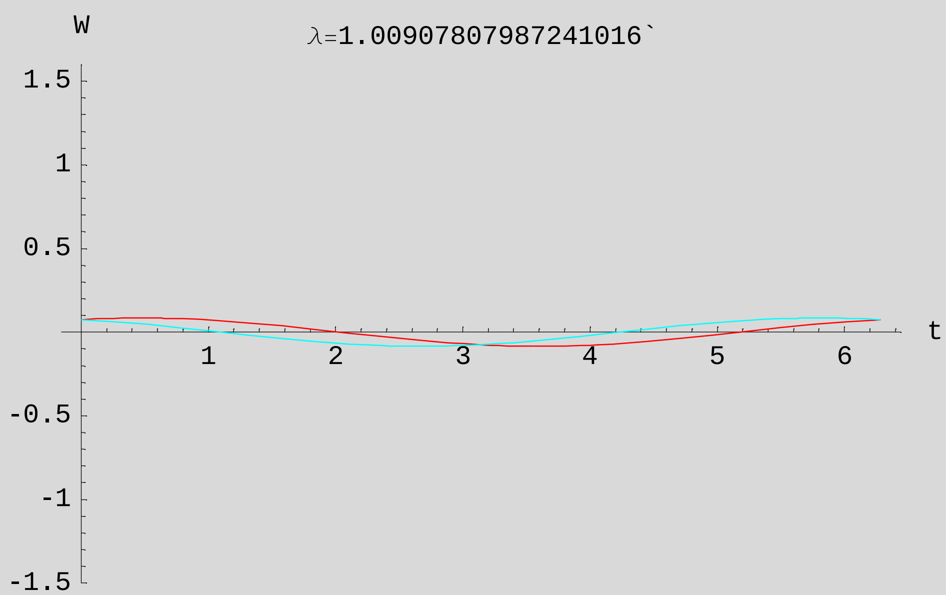
```
Length[bran1]
```

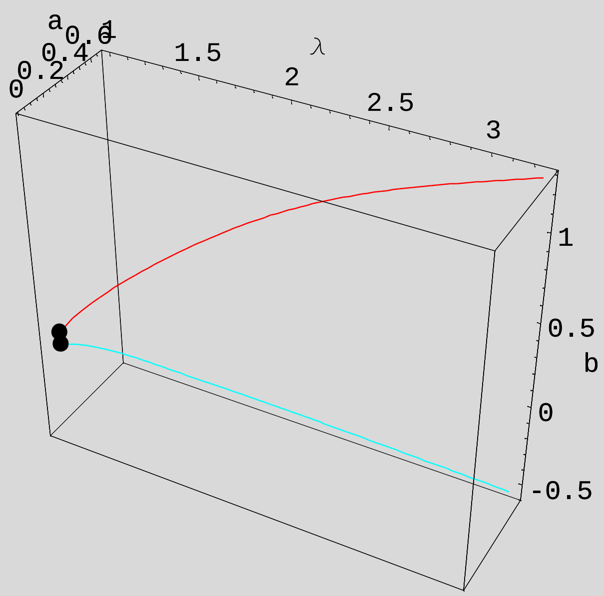
```
450
```

```
Length[bran2]
```

```
450
```

```
animatesolnbranches[1, 450, 28]
```






```
{ GraphicsArray , GraphicsArray ,
  GraphicsArray , GraphicsArray , GraphicsArray ,
  GraphicsArray , GraphicsArray , GraphicsArray ,
  GraphicsArray , GraphicsArray , GraphicsArray ,
  GraphicsArray , GraphicsArray , GraphicsArray ,
  GraphicsArray , GraphicsArray , GraphicsArray }
```

4. $L = \nabla^2$, $H = C^2[(0, 2\ \ )\times(0, 2\ \ )]$.

In this section we will look at examples of the form

```
∇² w + λ w + f[x, y, w, ∂x w, ∂y w] == 0,
```

with the boundary conditions,

```
w[0, y] == w[2 π] == w[x, 0] == w[x, 2 π] == 0
```

Here $\nabla^2 = \ _{x,x} + \ _{y,y}$.

It is not hard to see that the **linear boundary-value problem**

```
∇² w + λ w == 0
```

(with the same boundary conditions), has the eigenvalues $\lambda = \lambda_{m\,n} = m^2 + n^2$,with **m, n** being positive integers. The corresponding eigenfunctions are
$w_{m\,n}$**[x,y] = Sin[m x] Sin[n y].**

We will be interested in bifurcation from the **double eigenvalue** $\lambda_{1\,3}$ **= 10**. The corresponding eigenfunctions are **Sin[x] Sin[3 y]** and **Sin[3 x] Sin[y].** Therefore **dim N** = **2** and **dim** $\mathbf{N}^{\perp}$ **is .** We will take a basis for **N** to be
{Sin[x] Sin[3 y], Sin[3 x] Sin[y]}.

We will use the inner product

```
⟨f_, g_⟩ := ∫_0^{2π} ( ∫_0^{2π} f[x, y] g[x, y] dx ) dy
```

Therefore the vectors in the unit sphere(circle) in **N** can then be parametrized using the angle α **in [0 , 2)** by

```
w[α_][x_, y_] :=
  1/π (Cos[α] Sin[x] Sin[3 y] + Sin[α] Sin[3 x] Sin[y])
```

i.e.,

```
⟨w[α], w[α]⟩
```

It is easy to see that

```
wp[α_][x_, y_] :=
  1/π (Sin[α] Sin[x] Sin[3 y] - Cos[α] Sin[3 x] Sin[y])
```

is orthogonal to **w[α_][x_, y_]** and so is in the tangent plane(line) to the unit sphere(circle) in **N** at **w[α_][x_, y_]**.

i.e.,

```
⟨wp[α], w[α]⟩
```

4.1 An Example Where Bifurcation Does Occur.

For this example we take the nonlinear term to be,

```
f[x_, y_, w_, u_, v_] := x y w v^2
```

with

```
f[α_][x_, y_] =
 f[x, y, w, u, v] /. {w -> w[α][x, y], v -> ∂y w[α][x, y]}
```

Equation (6) then becomes

```
⟨wp[α], f[α]⟩
```

This integration takes an extremely long time. Performing the multiplication first and then integrating each term separately makes this calculation very fast (from many hours to seconds!), i.e.,

```
integrand = Expand[wp[α][x, y] f[α][x, y]]
```

```
Length[integrand]
```

```
12
```

$$\sum_{j=1}^{12} \int_0^{2\pi} \left(\int_0^{2\pi} \texttt{integrand[[j]]} \, dx \right) dy$$

```
Plot[%, {α, 0, 2π}]
```

This plot shows that there are eight transverse zeros on [0, 2). Therefore from the remark about tranverse zeros in Section 1, we conclude that there will be eight branches of solutions bifurcating from the eigenvalue $\lambda = 10$.

In this paper we will not construct these branches.

4.2 An Example Where Bifurcation Does Not Occur.

In general, bifurcation does occur. In order to construct an example where bifurcation does not occur, we had to use many of the symbolic capabilities of *Mathematica*. This example is the following. As in the previous example, we investigate the bifurcation from the eigenvalue

$_{13}$ **= 10**.

We take the nonlinear term to be,

```
f[x_, y_, w_, u_, v_] := h[x, y] w v^2
```

where,

$$h[x\_, y\_] = -\frac{98370678784\,\mathrm{Cos}[3\,y]^2\,\mathrm{Sin}[x]^2\,\mathrm{Sin}[3\,x]^2\,\mathrm{Sin}[y]^2}{306168435\,\pi^2} + \frac{17973345280\,\mathrm{Cos}[y]\,\mathrm{Cos}[3\,y]\,\mathrm{Sin}[x]\,\mathrm{Sin}[3\,x]^3\,\mathrm{Sin}[y]^2}{61233687\,\pi^2} + \frac{288563234816\,\mathrm{Cos}[y]^2\,\mathrm{Sin}[3\,x]^4\,\mathrm{Sin}[y]^2}{306168435\,\pi^2} + \frac{4213907456\,\mathrm{Cos}[3\,y]^2\,\mathrm{Sin}[x]^3\,\mathrm{Sin}[3\,x]\,\mathrm{Sin}[y]\,\mathrm{Sin}[3\,y]}{61233687\,\pi^2} + \frac{1}{34018715\,\pi^2}(14546558976\,\mathrm{Cos}[y]\,\mathrm{Cos}[3\,y]\,\mathrm{Sin}[x]^2\,\mathrm{Sin}[3\,x]^2\,\mathrm{Sin}[y]\,\mathrm{Sin}[3\,y]);$$

```
f[α_][x_, y_] =
  f[x, y, w, u, v] /. {w -> w[α][x, y], v -> ∂_y w[α][x, y]};
```

Equation (6) then becomes

```
⟨wp[α], f[α]⟩
```

As in the previous example this integration takes an extremely long time. **Performing the multplication first and then integrating each term separately makes this calculation very fast**, **i.e.,**

```
integrand = Expand[wp[α][x, y] f[α][x, y]]
```

```
Length[integrand]
```

```
48
```

$$\sum_{j=1}^{48} \int_0^{2\pi} \left(\int_0^{2\pi} \mathrm{integrand}[[j]]\, dx\right) dy$$

$$\frac{\mathrm{Cos}[\alpha]^4}{\pi^4} + \frac{\mathrm{Sin}[\alpha]^4}{\pi^4}$$

This function has no zeros. Therefore from Theorem 1 in Section 1, we conclude that there will be no bifurcation from the eigenvalue = 10.

5. References

[1] Wolkowisky, J.H., A Geometric Theory of Bifurcation, *Proceedings of Symposia in Pure Mathematics***, 45, part 2 (1986), 553-564.**
[2] Wolkowisky, J.H., Bifurcation and Even-Like Vector Fields, *Proceedings of Symposia in Pure Mathematics*, 45, part 2 (1986), 331-340.