The Structure of Economic Models : Theory of Graphs with Mathematica

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One may have difficulties to analyse economic models even with those containing a few number of equations. For large macroeconometric models, where the equations are often non−linear, a block triangular boolean 0−1 matrix will tell us how to organize the whole system of equations and will be helpful for recursive resolutions. Furthermore, the knowledge of the structural properties may show obvious but also unsuspected properties such that a dominant or dominated position, even within a set of interdependant variables. Then the degree of eccentricity of some component will be considered. This problematic may be approached by means of oriented graphs and their algorithms, as it was proposed earlier by us [8, 9] in the 1970's.
In this paper, we are convinced of the major interest of using oriented digraphs (without loops) for analysing both theoretical and econometric models. In this paper, we are doing three main proposals for graph applications : the construction of a more performant graph representation using a largest circuit (after an exhaustive enumeration of circuits), the search after edge−disjoint cycles and the elaboration of a typology of the vertices . The application will be the dynamic Klein−Goldberger model [3, 10] with 26 equations. The computations are done with *Mathematica* 5.1 [14,15] and specialized packages for graphs, such as `DecisionAnalysis‘Combinatorica` [11] and `DiscreteMath‘GraphPlot` which documentation can be found at `http:/library.wolfram.com/infocenter/`. We have also used our own `Mathlink`-compatible external programs written in `Fortran F77L` [1,2] such as `"Baobab"` . This program returns an exhaustive list of lexicographically ordered circuits.

■ 1. The Klein − Goldberger Macroeconometric Model

The Klein–Goldberger model for the United States [3,10] (henceforth KG–model in the text) has been the essential reference in earlier econometric modelling which mainly purposes are forecasting and simulation of economic policies as well [4]. Many analysis have been achieved with this small size keynesian model to discover its static and dynamic properties [3,10]. In this section we will present the containts of the model and the organization of its equations.

□ 1.1. Presentation

An economic model with n *endogeneous* (X_i, i =1,n) and m *exogeneous* (Z_k , k = 0, m – 1) variables where Z_0 is a constant, can be described by the following equations system

$$X_i = f_i(X_1, ..., X_{i-1}, X_{i+1}, ..., X_n, Z_0, Z_1 ..., Z_m), \ i, = 1, n,$$

In the following list, we present the 26 endogeneous of the model. We indicate first the number of the equation that determines only one variable, the name of the variable and an approximative but more readible definition. In the KG–model, we have the following endogeneous variables

1. C : real consumer expenditures; **2. I** : real investment;
3. Sp : deflated corporate savings; **4. Pc** : deflated corporate profits;
5. D : real capital consumption charges;
6. W1 : deflated private employee compensation; **7. Nw** : number of salaries;
8. w : worked hours; **9. Fi** : imports; **10. A1** : deflated farm income;
11. pa : agricultural prices; **12. L1** : deflated liquid assets held by persons;
13. L2 : deflated liquid assets held by businesses;
14. iL : average yield on corporate bonds;
15. iS : average yield on short term commercial paper;
16. Y + T + D : Gross National Product;
17. P : deflated nonwage nonfarm income; **18. p** : inflation;
19. K : real stock of capital; **20. B** : deflated corporate profits;
21. T : deflated net indirect taxes; **22. Tw** : deflated net taxes on wage income;
23. Tc : deflated corporate income taxes; **24. Tp** : deflated net taxes on non wage;
25. Ta : deflated net tax es on farm income; **26. Y** : domestic production.

Theoretically the KG–model is a keynesian dynamic IS–LM model of an open economy, where the demand of goods and services plays an essential role in the production determination. To associate a graph to this model, the economist must say how to "read" the model : there may be different ways to associate the endogeneous variable to one equation of a model. A model may have different theoretical explanations. This problem is known under the graph–theoretic term a maximal bipartite matching. . A matching in the bipartite graph g =(X,Y,e) matches each vertex in X (the set of variables) to one in Y (the set of equations). . The primitive AddEdges[g,e] of the package Combinatorica returns a graph g(X,Y,e) . An adequate test for that graph is given by BipartiteQ[g]. The ranked embedded graph is rendered by RankedEmbedding[g,t], where parameter t numbers the vertices from 1 to 26 for the variables and from 27 to 52 for the equations. Finally, we may highlight edges that precise what association can be retained. The complete set of primitives is given by g=AddEdges[EmptyGraph[26,Type→Directed],{edges}], where the term "edges" represents pairs like {x1,y1}, {x3,y1},(x2,y2),... such that equation y1 can determine variables x1 or x3 and where equation y2 determine x2, ShowGraph[SetGraphOptions[RankedEmbedding[g,t],{matching},options]], where matching represents the list of edges realizing the property of a matching in a bipartite graph. To load the package Combinatorica evaluate

```
<< DiscreteMath`Combinatorica`
```

□ 1.2. Block Triangularity

The 26−vertex graph is displayed by the primitive g = AddEdges[EmptyGraph[26,Type−>Directed],{edges}]] where the adjacency list of directed edges takes the form {{x,y},....} when x −> y. In the static version of the KG−model the graph is composed of 26 vertices and 68 edges. The adjacency 0−1 matrix of the graph g has 26^2 entries. It is achieved by using the the command ToAdjacencyMatrix[g]// TableForm. The adjacency matrix of the KG−model model shows a scarce matrix. The Figure 1. (right) shows how a block triangularity can be achieved.

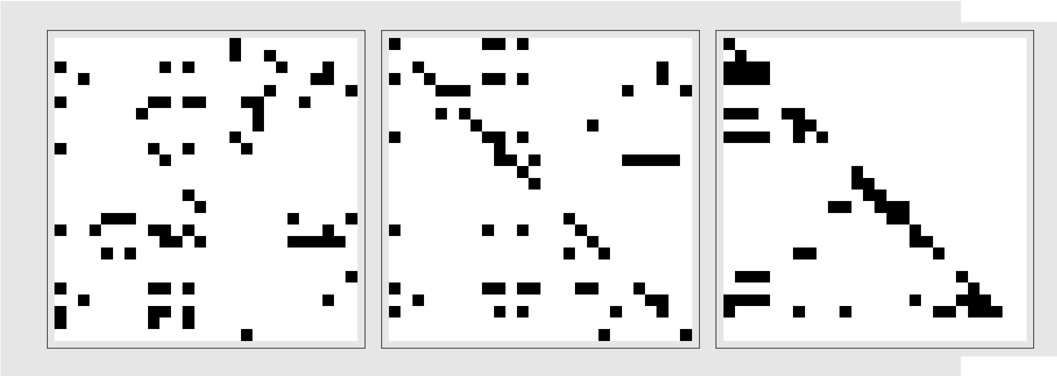

Figure 1. The block triangularity of the Klein−Goldberger model in static version. The figures illustrate the steps of the procedure. The figure (left) shows a boolean 0−1 scarce adjacency matrix . These tables have been obtained with help of the primitives of the DicreteMath'GraphPlot package. In this package, the initial matrix is displayed by the primitive ArrayPlot[ind, ColorsRules−>{0−>White,1−>Black}]. The primitive StrongComponents[ind] finds the strong connected vertices, This orders the adjacency matrix by symetrically permuting rows and columns. The figure (middle) shows a block triangular matrix with as many non zero entries on the diagonal as possible.Two primitives are used to make the matrix block triangular : MaximalBipartiteMatching[g] to permute entries to the diagonal, and StrongComponents[mat].

The commands of the program DiscreteMath'GraphPlot are coming from the Mathematica 5.1 Documentation

http://documents.wolfram.com/mathematica/Add−OnsLinks/StandardPackages/DiscreteMath. The primitives of the Figure1. (middle) are

```
ind = MaximalBipartiteMatching[spar];
   pind = SparseArray[{ind → Table[1, {Length[ind]}]}, {7, 7}];
      pind = Transpose[pind]; mat2 = pind . spar;
         figure2 = ArrayPlot[mat2, ColorRules → {0 → White, 1 → Black},
                     TextStyle → {FontSize → 12, FontFamily −> "Times"}]
```

The primitives of Figure 1. (right) are

```
perm = StrongComponents[mat2];
 perm2 = Flatten[perm];      func = mat2[[perm2, perm2]];
  figure3 = ArrayPlot[func, ColorRules → {0 → White, 1 → Black},
                    PlotLabel −> "(c)  block triangular  0−1 matrix",
                        TextStyle → {FontSize → 12, FontFamily −> "Times"}]
```

The 26− equations of the model can be organized according to that structure and solved recursively.

2. Structural Analysis of the Klein − Goldberger Model

We will also use specialized packages for drawing graphs such as DecisionAnalysis‘Combinatorica , DiscreteMath‘GraphPlot and our own Mathlink−compatible external programs written in Fortran F77L "Baobab" for the enumeration of circuits. The Section 2.2 deals with the connectivity of the graph .We will examine two versions of the model : a "static" one where the exogeneous (instantaeous and delayed) and the delayed endogeneous variables have been replaced by their observed values; the other is" dynamic" since the delayed endogeneous variables are taken into account.

2.1. Graph Representation

The 26−vertex graph of the KG−model is displayed by the primitive g = AddEdges[EmptyGraph[26,Type−>Directed],{edges}]]. The whole graph is shown in the right Figure 2.(left) (section 2.2) . All vertices are organized around a circle. In this application, we use the primitives

```
ShowGraph[g,
  VertexNumber → True, VertexColor → Blue,
  TextStyle → {FontSize → 12, FontColor → Red, FontFamily −> "Times"},
    HeadCenter → .9, HeadWidth → .25,
      Background → LightBlue, AspectRatio → 1.]
```

Our proposal may be described as follows : firstly identify the largest strong connected components, secondly search for the (or one) largest circuit and place the vertices of the largest circuit on a polygon, thirdly put (inside the polygon) the remaining vertices of the component , introduce the remaining vertices outside and finally rely all the pairs of vertices by oriented edges (arrows) according to the information given by the graph. A polygonal curve is created by the primitives of *Mathematica* with data = Table[{Cos[n 2 Pi/20],Sin[n 2 Pi/20}],{n,0,20}]; Polygon[data]; InputForm[%]; Show[%,options] . The arrows between vertices are then drawn by using the primitives of the Standard Add−on package Graphics‘Arrow , Thus, we have to use the function Arrow[{x,y},{u,v}] for P(x,y) −>Q(u,v). and Show[Graphics[{Arrow[{x,y},{u,v}] ,Arrow[{x,y},{r,s}],...}]] to draw the entire graph. The Figure 2 (middle) shows a more readible graph. In this figure the graph is restricted to the strong connected component.

□ 2.2. Strongly Connected Components

The Figure 2. (left) shows the initial graph by Combinatorica , the largest strong connected component in Figure 2. (middle) and the reduced graph of the model in Figure 2 (right). We can soon deduce two main observations : the model is largely interdependant with in the graph a strong connected component composed of 20 vertices, the reduced graph emphazises the keynesian character of that model with a dominant investment, interest rates (short term and long term) that affect mainly the liquid assets held by agents together with the interdependant variables.

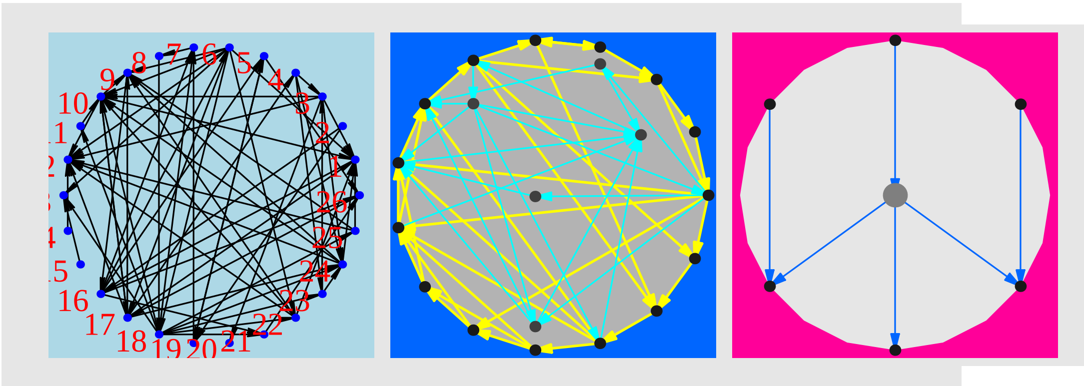

Figure 2. Graph representations of the static KG−model. The figure (left) has been drawn with help of the command ShowGraph[g ,options] of the package Combinatorica. The figure (middle) shows the largest strong connected component. The presentation is that we preconize using the primitives of Mathematica and those of the Standard Add−on package Graphics‘Arrow . The figure (right) represents the whole graph in a reduced form without circuits. In this figure, such that the strong connected component has been contracted to a single vertex. The structure is conformed to the keysesian theory of short term. The variable investment dominates , benefits of firms have an output position, and interest rates affect the liquid assets held by persons and businesses.

The contracted graph is obtained by using the command ShowGraphArray[{g,Contract[g,{list of vertices to delete}]},VertexNumber−>True] .The primitive Contract has some effects : this command maintains the embedding of the initial graph, deleting a vertex it deletes all incident edges, a new vertex is placed ar the midpoint of the contracted vertices, and the vertices are automatically renumbered.

The Figure 3. show how the structure of the modified when we consider the delayed endogeneous variables (the dynamic KG−model). Sure, the density of relations is increased in the initial graph. The largest strong connected component includes now variables that were "external" in the static version such as : investment, profit and liabilities. Interest rates are now dominant variables.

The comparison between Figures 2. (static version) and Figure 3. (dynamic version) shows expected results for such a model : investment and corporate benefits become interdependant since they embed the strong connected component.

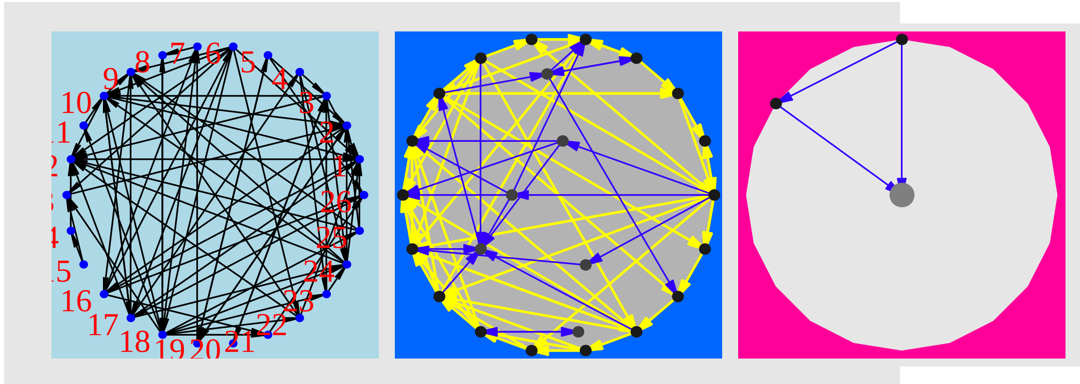

Figure 3. Graph representations of the dynamic KG model. The figures for the dynamic version show some notable modifications. The strongly connected component (the largest) is more important and integrates investment and benefits of firms . Interest rates dominates in the figure (right).

□ 2.3. Lexicographically Ordered Circuits

The Figure 4. shows the repartition of the circuits for both static and dynamic versions of the KG–model with two presentations, a bart chart and a pie chart. The numbers and distributions of the circuits are different : more circuits (=586) in dynamic version in comparison to a 255 circuits in the static KG–model, more important length of the largest circuit in dynamic version, with 18 vertices against 15 for the static version, greater dispersion of length of circuits in the dynamic version.

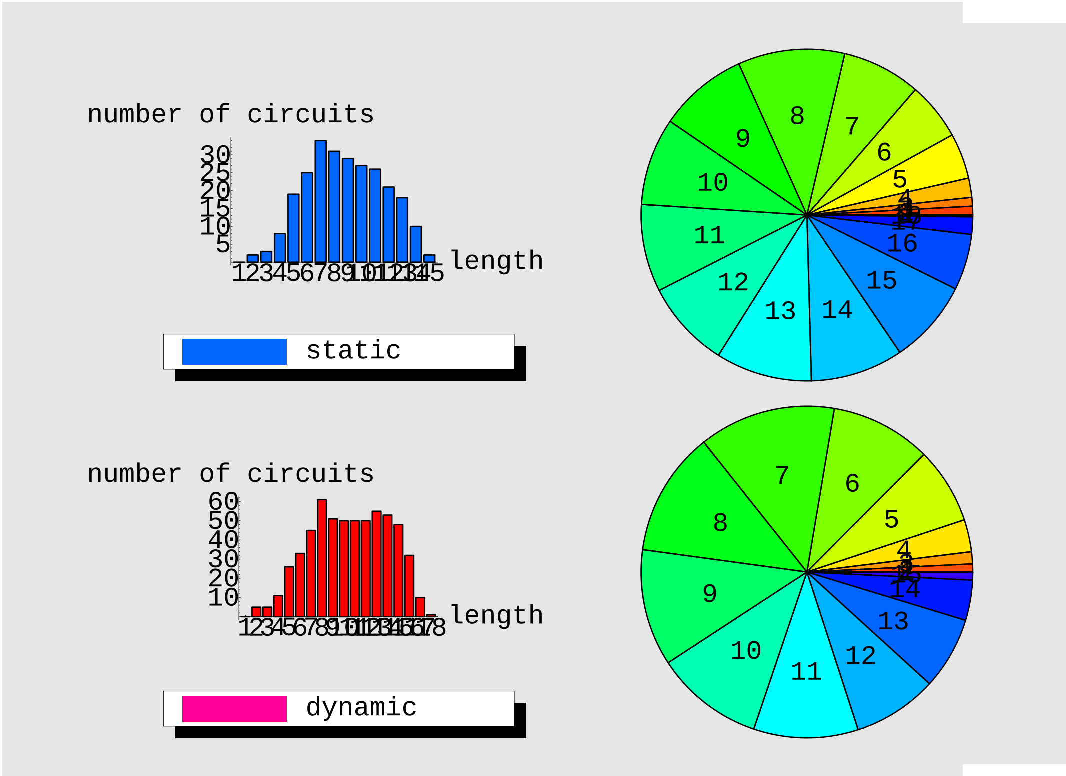

Figure 4. Lexicographic ordered circuits. Our program "Baobab" gives an exhaustive list of circuits and classifies them according to an lexicographic order. We distinguish the static and the dynamic version of the KG−model. For each version, we produce a barchart using the command BarChart[data,options] of the Standard Add−on Package Graphics'Graphics and Graphics'Legend. We use the command in the following way : figure = BartChart[datalist, options]; ShowLegend[Show[figure],{{"text"}},options}]. Pie charts are created by the simple command PieChart[datalist,options] . The number of circuits in the dynamic version exceeds that one of the static version with 586 circuits (against 255 in the static KG−model) and with a largest length ot 18 vertices in dynamic version (against 15 in the static KG−model).

2.4. Edge−Disjoint Cycles

When investigating all the circuits, we may be interested in the circuits that have no edge in common. The primitive ExtractCycles[g] of the package Combinatorica determines these "independant circuits". In Figure 5. we have highlighted the extracted circuits with different colors.

In the static KG−model (255 circuits) four "independant" circuits are observed . These circles has been colored differently. The figures with a lightblue background are those of the static KG−model and the other (right) that one of the dynamic KG−model. The Figure 5. (middle) shows an improvement in the presentation., when the position of one vertex is modified. The following results give the economic signification of such Figure 5. (middle). The color of the text corresponds to that of the circuits . The four "independant" circuits of the static version can be interpretated as follows with the number and the abreviation of the variables (see also the table of definitions in Section 1.1). The circuit **{ 19, 5, 19} ≡ {K, D, K}** represents real capital accumulation in the Figure 5 (middle) , the circuit **{9, 16 6, 9} ≡ {Fi, YTD, W_1 ,Fi}** between the national product and importations, the circuit **{3, 10, 17, 4, 3}≡ {S_p , A_1 , P, P_c , S_p }** represents the repartition of the revenues, and the circuit **{17, 1, 16, 5, 26, 17} ≡ {P, C, YTD, D, Y, P}** is the circuit of the national revenue.

In the dynamic KG−model (586 circuits) we have eight "independant" circuits which interpretation is : **{17, 10, 17} ≡ {P, A_1 ,P}** a circuit between farm and nonfarm incomes, the circuit **{18,8,18} ≡ {p,w,p}** is the Phillips curve , the circuit **{19, 2, 19} ≡ {K, I, K}** for investment, the circuit **{20, 3,20} ≡ {B, S_p ,B}** between the corporate and savings, **{6,1,16,6} ≡ {W_1 ,C,YTD, W_1 }** a dark blue circuit for consumption, the circuit **{10,2, 16, 7, 18,10} ≡ {A_1 , I,TTD, N_w , p, A_1 }** a purple circuit for investment demand , inflation and revenues, and the largest circuit of the revenue **{4, 3, 10, 9, 16, 21, 26, 17,4} ≡ {P_c , S_P , A_1 , Fi, YTD, T, Y, P, P_C }.**

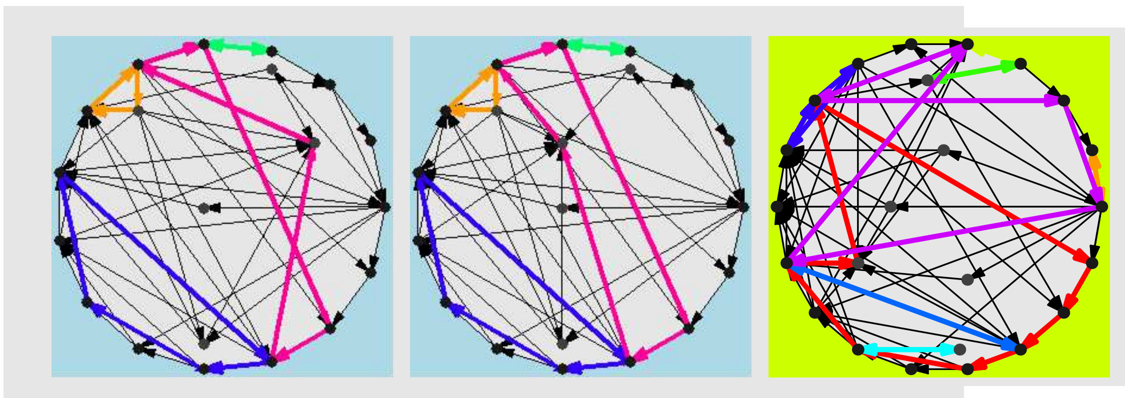

Figure 5. Maximal lists of edge−disjoint cycles in the static and dynamic KG−model. The figures (left) and (middle) concern the static version. We show the interest of extracting the edge−disjoint cycles, where no cycle of the list have an edge in common. The information of figure (left) help for improving the graph representation. As concerns the dynamic KG−Model, we indicate also what circuits we find by using ExtractCycles[g]. In this version the "basis" of circuits has a higher dimension. The economic meaning of such circuits is given in the text.

■ 3. Typology of variables in Klein − Goldberger Model

This Section proposes a classification of the variables that is based on the structural properties of the two versions of the KG−model.

□ 3.1. Largest All−pairs shortest−path matrix

A computation of the all−pairs shortest−path matrix and search for its properties will be realized on the two versions of the KG−model.

The Static Klein−Goldberger Model

This computation of the largest all−pairs shortest−path matrix is essential to appreciate the eccentricity of different vertices. From this matrix, we may then explore the maximum of each row. This matrix will be bordered by a column vector containing these maximum values for each row. Each value represents the maximum of the shortest path from one vertex to each every other vertices. Eccentricity will then corresponds to the longest shortest path from one vertex to all other vertices. The maximum eccentricity is the diameter and we will look for the variables that possess this property. The minimum eccentricity is the radius of that graph and we will also indicate what variables have this property. The all−pairs shortest−path are vieved by using by the command AllPairsShortestPath[w]//TableForm. The largest all−pairs shortest−path matrix for the static version is given in Table 1.

	C	Sp	Pc	D	W1	Nw	w	Fi	A1	Pa	YTD	P	p	K	T	Tw	Tc	Tp	Ta	Y
C	0	5	4	2	2	2	3	3	3	4	1	3	3	3	2	3	4	4	4	2
Sp	1	0	3	3	3	3	4	2	1	5	2	2	4	4	3	4	4	1	5	3
Pc	2	1	0	4	4	4	5	2	2	6	3	3	5	5	4	5	1	1	6	4
D	3	4	3	0	5	2	3	3	3	4	4	2	3	1	4	4	4	3	4	1
W1	1	3	2	3	0	3	4	1	1	2	2	1	1	4	2	1	2	2	2	3
Nw	3	3	4	5	5	0	1	3	2	2	4	3	1	6	2	2	2	2	2	3
w	3	3	4	5	5	5	0	3	2	2	4	3	1	6	2	2	2	2	2	3
Fi	3	5	4	2	2	2	3	0	3	4	1	3	3	3	2	3	4	4	4	2
A1	1	3	2	3	3	3	4	1	0	5	2	1	4	4	3	4	3	2	5	3
Pa	2	4	3	4	4	4	5	2	1	0	3	2	5	5	4	5	4	3	6	4
YTD	2	4	3	1	1	1	2	2	2	3	0	2	2	2	1	2	3	3	3	1
P	1	2	1	3	3	3	4	1	1	5	2	0	4	4	3	4	2	1	5	3
p	2	2	3	4	4	4	5	2	1	1	3	2	0	5	1	1	1	1	1	2
K	4	4	4	1	6	1	2	4	3	3	5	3	2	0	3	3	3	3	3	2
T	3	4	3	5	5	5	6	3	3	7	4	2	6	6	0	6	4	3	7	1
Tw	1	4	3	3	3	3	4	1	1	5	2	2	4	4	3	0	4	3	5	3
Tc	2	1	4	4	4	4	5	2	2	6	3	3	5	5	4	5	0	1	6	4
Tp	1	4	3	3	3	3	4	1	1	5	2	2	4	4	3	4	4	0	5	3
Ta	1	6	5	3	3	3	4	1	4	5	2	4	4	4	3	4	5	5	0	3
Y	2	3	2	4	4	4	5	2	2	6	3	1	5	5	4	5	3	2	6	0

Table 1. Largest all−pairs shortest−path matrix in the static version of the Klein − Goldberger model. A single−source shortest−path consists of finding the shortest paths between one given sources to all other vertices. The all−pairs shortest−paths will then be deduced by considering each vertex as a source successively. The table gives such results for the sub−graph which is the largest strong connected component. For example (C,W1) = 2 signifies that the shortest path going from the source C and the the sink W1 has two edges. We also have (w,Pc) = 4 and (Pc,w) = 5. We use the primitives of the package Combinatorica with scc = DeteteVertices[g,list of vertices]; AllPairsShortestPath[scc] // MatrixForm. The command DeleteVertices[g,{7,17}] deletes the vertices 7 and 17 from graph g, and all incident edges.To produce the table, the set of commands are of the type : FrameBox[StyleForm[TableForm[s,TableHeadings−>{list,list}], options] // DisPlayForm.

The maxima of the shortest paths are given in a vector column **{5,5,6,5,4,6,6,5,5,6,4,5,5,6,7,5,6,5,6,6}**, that must be placed to the right to the matrix. The second element of that list indicates that the maximum distance from **2 : Sp** is five edges.From this list, we deduce that the radius equals 4 and the diameter equals 7. These results are obtained by the following directives : Eccentricity[g] gives the eccentricity of each vertex of graph g, GraphCenter[g] gives the list of of the vertices of graph g with minimum eccentricity, {Diameter[g], Radius[g]} where Diameter[g] is the maximum length among all pairs of vertices in g and where Radius[g] returns the minimum eccentricity of any vertex of g. The static KG−model has two centers **6 : W1** (private compensation), **16 : GNP** and one eccentric variable **21 : T** (indirect taxes). In this model production and repartition play a central role. One fiscality variable has a weak influence unless it is interdependant.

From Table 1, we can also deduce the "perturbation" or anti−eccentricity that each vertex is receiving from the other vertices. In this case we have to explore the columns of the all−pair shortest−path matrix. The maxima of shortest paths are given in a row vector **{4,6,5,5,6,5,6,4,4,7,5,4,6,6,4,6,5,5,7,4}**, that must be placed below the matrix. The anti−centers are **1 : C** (consumption), **9 : Fi** (import) , **10 : A1** (farm income), **17 : P** (non wga income) , **21 : T** (indirect taxes) , **26 : Y** (production). The ant−eccentric vertices are : **11 : pa** (prices in agriculture), **25 : Ta** (farm income).

The dynamic Klein−Goldberger Model

The largest all−pairs shortest−path matrix in the dynamic version of the Klein − Goldberger model is shown in Table 2.

	C	I	Sp	Pc	D	W1	Nw	w	Fi	A1	Pa	L1	L2	YTD	P	p	K	B	T	Tw	Tc	Tp	Ta	Y
C	0	4	5	4	6	2	2	3	3	3	4	3	3	1	3	3	5	6	2	3	4	4	4	2
I	3	0	5	4	2	2	2	3	3	3	4	3	3	1	3	3	1	6	2	3	4	4	4	2
Sp	1	2	0	3	4	3	3	4	2	1	5	1	4	2	2	4	3	1	3	4	4	1	5	3
Pc	2	2	1	0	4	4	4	5	2	2	6	2	5	3	3	5	3	2	4	5	1	1	6	4
D	3	1	4	3	0	3	2	3	3	3	4	3	4	2	2	3	1	5	3	4	4	3	4	1
W1	1	2	3	2	4	0	3	2	1	1	2	1	1	2	1	1	3	4	2	1	2	2	2	3
Nw	3	3	3	4	5	5	0	1	3	2	2	3	2	4	3	1	4	4	2	2	2	2	2	3
w	3	3	3	4	5	5	5	0	3	2	2	3	2	4	3	1	4	4	2	2	2	2	2	3
Fi	3	4	5	4	6	2	2	3	0	3	4	3	3	1	3	3	5	6	2	3	4	4	4	2
A1	1	1	3	2	3	3	3	4	1	0	5	1	4	2	1	4	2	4	3	4	3	2	5	3
Pa	2	2	4	3	4	4	4	5	2	1	0	2	5	3	2	5	3	5	4	5	4	3	6	4
L1	1	5	6	5	7	3	3	4	4	4	5	0	4	2	4	4	6	7	3	4	5	5	5	3
L2	4	1	6	5	3	3	3	4	4	4	5	4	0	2	4	4	2	7	3	4	5	5	5	3
YTD	2	3	4	3	5	1	1	2	2	2	3	2	2	0	2	2	4	5	1	2	3	3	3	1
P	1	2	2	1	4	3	3	4	1	1	5	1	4	2	0	4	3	3	3	4	2	1	5	3
p	2	2	2	3	4	4	4	1	2	1	1	2	1	3	2	0	3	3	1	1	1	1	1	2
K	4	1	4	4	1	3	1	2	4	3	3	4	3	2	3	2	0	5	3	3	3	3	3	2
B	2	3	1	4	5	4	4	5	3	2	6	2	5	3	3	5	4	0	4	5	5	2	6	4
T	3	4	4	3	6	5	5	6	3	3	7	3	6	4	2	6	5	5	0	6	4	3	7	1
Tw	1	5	6	5	7	3	3	4	1	4	5	1	4	2	4	4	6	7	3	0	5	5	5	3
Tc	2	2	1	4	4	4	4	5	2	2	6	2	5	3	3	5	3	2	4	5	0	1	6	4
Tp	1	1	4	3	3	3	3	4	1	1	5	1	4	2	2	4	2	5	3	4	4	0	5	3
Ta	1	1	6	5	3	3	3	4	1	4	5	1	4	2	4	4	2	7	3	4	5	5	0	3
Y	2	3	3	2	5	4	4	5	2	2	6	2	5	3	1	5	4	4	4	5	3	2	6	0

Table 2. Largest all−pairs shortest−path matrix in the dynamic version of the Klein − Goldberger model. The matrix has the same signification as above. In the dynamic version we have the same results for these examples : (C,W1) = 2 , (w,Pc) = 4 and (Pc,w) = 5.

The maxima of the shortest paths are given in a vector column **{6,6,5,6,5,4,5,5,6,5,6,7,7,5,5,4,5,6,7,7,6,5,7,6}**. From this list,we deduce that the radius equals 4 and the diameter equals 7. The dynamic KG−model has two centers **6 : W1** (private compensation), **8 : w** (worked hours) and five eccentric variables such as **12 : L1** (liquid assets held by persons), **13 : L2** (liquid assets held by businesses), **21 : T** (indirect taxes), **22 : Tw** (taxes on wage income), **25 : Ta** (taxes on farm income). The variables of revenue and worked hours exert a great influence on the rest of the model. The monetary variables (liquidities of persons and firms) are eccentric and exert less influences. In that sense , this is conformed to the theory.

From Table 2., we can also appreciate the "perturbation" or anti−eccentricity that each vertex is receiving from the other vertices. In this case we have to explore the columns of the largest all−pair shortest−path matrix. The maxima of shortest paths are given in a row vector **{4,6,5,5,6,5,6,4,4,7,5,4,6,6,4,6,5,5,7,4}**. The anti−center are **1 : C** (consomption), **9 : Fi** (Import) , **10 : A1** (farm income), **17 : P** (nonwage income) **, 21 : T** (indirect taxes) , **26 : Y** (production). The anti−eccentric vertices are : **11 : pa** (prices in agriculture), **25 : Ta** (taxes on farm income). Thus the total supply (production and import), the consumer demand and indirect taxes, nonwage incomes are rather pertubed by the rest of the model. Agricultural prices and taxes are less influenced.

□ 3.2. Typology of Vertices

In Section 3.1, we have shown several structural propertiess for the vertices (or variables) that were based on the concepts of eccentricity and anti−eccentricity. The exploitation of the bordered all−pairs shortest−path matrix introduced the properties for a vertex to be a center, an eccentric vertex or else (column vertor) and also to anti−center, be anti−eccentric or else.

Our simple approach consists of crossing the preceeding properties into a table that will collect the results (see highlighted tables below). The tables have three columns and three rows. The colums (from left to right) represent each one a particular property : "eccentric","center" and"else". The rows (from top to bottom) represent each of them one particular property : "anti−eccentric","anti−center" and"else". Crossing these two aspects, we then define different types for the variables. The **type A** describes an interdependant variable which is "eccentric" and "anti−eccentric" : this variable will then have little influence and will be less perturbated. The **type B** corresponds to a variable which is weak "dominant" as a center but an anti−eccentric variable (**type B*** means strong dominant**). The type C** is a weak "integrated " variable as it diffusse rapidely and is rapidely perturbed (**type C*** means strong integrated). The **type D** figures a weak "dominated" variable since its influence is low and and its perturbation hight (**type D*** means strong dominated). The **type E** corresponds to variables without remarkable features. In the Table 2. we have highlighted the cases where we find variables of these type in th static KG−model).

According to Table 3. the set of interdependant variables is composed of weak integrated variables (C) According to that table, the set of interdependant variables is composed of weak integrated variables (C), weak dominant and strong dominated variables (B and D*). The weak dominant variables are agricultural prices and net fiscality of farmers. The strong dominated variable is indirect taxes less transfers to agriculture. In the weak integrated variables, we find the national revenue and the production, the wages and non−wages , the consumption and the imports.

A	**B***	**B**
D*	**C***	**C**
D	**C**	**E**

Table 3. Typology of variables in static of the Klein − Goldberger model. According to that table, the set of interdependant variables is composed of weak integrated variables (C*), dominant and dominated variables (B* and D), and eccentric variables with low influence and perturbation.

To obtain this table we define rows as functions and columns as correspondances. The rows simply use the primitives of Mathematica : Plot[{1,2,3},{x,0,3}] for the functions and Graphics[{Line[{{2,3},{0,3}},...}] for drawing the correspondances. The Standard Add–on Package Graphics‘FilledPlot‘ is used to highlight the choosen area with a command like FilledPlot[{2,3},{x,1,2}, Fills→{GrayLevel[.8]},Curves−>Front,options].

The Table 4. shows the typology of the dynamic KG–model.

A	**B***	**B**
D*	**C***	**C**
D	**C**	**E**

Table 4. Typology of variables in the dynamic version of the Klein − Goldberger model. In this dynamic version, where the interdependant variables are more numerous, we find just integrated variables (in C*) and dominant ones in B* .

According to Table 4., the set of interdependant variables is composed of weak integrated variables (C). We have the following characteristics : weaks integrated variables with gross national product, consumption, imports, prices and total incomes. The weak dominant variables are profits, agricultural price and depreciation on real capital. Liquidities and taxes are weak or strong dominated. An eccentric variable is net fiscality. If we compare both results in the static and in the dynamic version of the KG − model, we find a stability for the weak integrated variables, but new variables enter in the set of dominated variables.

4. Conclusions

The objective of this paper was restricted to the evaluation of the theory of graph for analysing economic models with help of the *Mathematica* 5.1 package. The reference macroeconometric Klein − Goldberger for USA has been choosen as application. The results may be summarized as follows : 1) associating a graph to a model requires one or more interpretations of the economist (for example a keynesian interpretation in short term, and a classical one in the long term for the same model). In terms of the theory of graphs it is a bipartite matching), 2) the graph representation may be improved by organizing the vertices of the largest strong connected component around the largest circuit , 3) the enumeration of the circuits and choice of circuits of maximal length is an useful computation, 4) a typology for the interdependant variables that is deduced from the structural properties gives an interesting insight inside a block with many interrelations. This application shows the performances of *Mathematica* 5.1 , of associated packages for graphs (such as `DecisionAnalysis‘Combinatorica, DiscreteMath‘GraphPlot)` and also `Mathlink−compatible external programs.`

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