On the Perimeter of an Ellipse

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Computing accurate approximations to the perimeter of an ellipse is a favourite problem of amateur mathematicians, even attracting luminaries such as Ramanujan [1, 2, 3]. As is well known, the perimeter, $P$, of an ellipse with semimajor axis $a$ and semiminor axis $b$ can be expressed exactly as a complete elliptic integral of the second kind, which can also be written as a Gaussian hypergeometric function,

$$P = 4aE\left(1 - \frac{b^2}{a^2}\right) = 2\pi a F_1\left(\frac{1}{2}, -\frac{1}{2}; 1; 1 - \frac{b^2}{a^2}\right).$$  \hspace{1cm} (1)

What is less well known is that the various exact forms attributed to Maclaurin, Gauss–Kummer, and Euler, are related via quadratic transformation formulae for hypergeometric functions. In this way we obtain additional identities, including a particularly elegant formula, symmetric in $a$ and $b$,

$$P = 2\pi \sqrt{ab} P_{\frac{1}{2}}\left(\frac{a^2 + b^2}{2ab}\right),$$ \hspace{1cm} (2)

where $P_{\frac{1}{2}}(z)$ is a Legendre function.

Approximate formulae can be obtained by truncating the series representations of exact formulas. For example, Kepler used the geometric mean, $P \approx 2\pi \sqrt{ab}$.

In this paper, we examine the properties of a number of approximate formulas, using series methods, polynomial interpolation, rational polynomial approximants, and minimax methods.

- Cartesian Equation

The Cartesian equation for an ellipse with centre at $(0, 0)$, semimajor axis $a$, and semiminor axis $b$ reads

$$E(x, y) = \left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1;$$

Introducing the parameter $\phi$ into the Cartesian coordinates, as $(x = a \sin(\phi), y = b \cos(\phi))$, one verifies that the ellipse equation is satisfied.
Simplify[$E(a \sin(\phi), b \cos(\phi))$]

True

■ Arclength

In general, the parametric arclength is defined by

$$L = \int_{\phi_1}^{\phi_2} \sqrt{\left(\frac{\partial x}{\partial \phi}\right)^2 + \left(\frac{\partial y}{\partial \phi}\right)^2} \, d\phi$$

(3)

The arclength of an ellipse as a function of the parameter $\phi$ is an (incomplete) elliptic integral of the second kind.

$$L(\phi) = \text{With}[[x = a \sin(\phi), y = b \cos(\phi)],$$

Simplify$[\int \sqrt{\left(\frac{\partial x}{\partial \phi}\right)^2 + \left(\frac{\partial y}{\partial \phi}\right)^2} \, d\phi, a > b \land 0 < \phi < \frac{\pi}{2}]$$

$$a E\left(\phi \mid 1 - \frac{b^2}{a^2}\right)$$

Since,

$L(0) = 0$

True

the arclength of the ellipse is

$$L(\phi) = a E(\phi \mid e^2)$$

(4)

where the eccentricity, $e$, is defined by

$$e(a_-, b_-) = \sqrt{1 - \frac{b^2}{a^2}}.$$ 

■ Perimeter

Since the parameter ranges over $0 \leq \phi \leq \pi/2$ for one quarter of the ellipse, the perimeter of the ellipse is

$$P_1(a_-, b_-) = 4 L\left(\frac{\pi}{2}\right)$$

$$4 a E\left(1 - \frac{b^2}{a^2}\right)$$

That is $P = 4 a E(e^2)$ where $E(m)$ is the complete elliptic integral of the second kind.

□ Alternative Expressions for the Perimeter

The above expression for the perimeter of the ellipse is unsymmetrical with respect to the parameters $a$ and $b$. This is “unphysical” in that both parameters, being lengths of the (major and minor) axes, should be on the same footing. We can expect that a symmetric formula, when truncated, will more accurately approximate the perimeter for both $a \geq b$ and $a \leq b$.

Noting that the complete elliptic integral is a gaussian hypergeometric function,
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A symmetric formula for the perimeter of an ellipse can be obtained from quadratic transformation formulæ for gaussian hypergeometric functions. For example, using \( P_1(a, b) = 2 \pi a \, _2F_1\left(\frac{1}{2}, -\frac{1}{2}; 1; e(a, b)^2\right) \)

one obtains Maclaurin’s 1742 formula (see [2])

\( P_2(a, b, c) = \pi (a + b) \, _2F_1\left(-\frac{1}{2}, \frac{1}{2}; 1; 1 - \frac{4ab}{(a+b)^2}\right) \)

first obtained by Ivory (1796), but known as the Gauss–Kummer series (see [2]).

Introducing the homogenous symmetric parameter \( h = \frac{(a-b)^2}{(a+b)^2} = 1 - \frac{4ab}{(a+b)^2} \), one has (c.f. mathworld.wolfram.com/Ellipse.html),

\[
\pi (a + b) \, _2F_1\left(-\frac{1}{2}, \frac{1}{2}; 1; h\right) \quad \text{// FunctionExpand // Simplify}
\]

\( 2(a + b) \left(2E(h) + (h - 1)K(h)\right) \)

Explicitly, the Gauss–Kummer series reads

\[
P_3(a, b, c) = \text{FullSimplify}[P_2(a, b) \text{// FunctionExpand, } a > b > 0]
\]

\[
4(a + b) E\left(1 - \frac{4ab}{(a+b)^2}\right) - \frac{8abK\left(1 - \frac{4ab}{(a+b)^2}\right)}{a + b}
\]

Instead, using functions.wolfram.com/07.23.17.0103.01, one obtains Euler’s 1773 formula (see also [2]):
\[ 2 F_1(a, \beta; 2 \beta; z) = \frac{2 F_1\left(\frac{a}{2}, \frac{a+1}{2}; \beta + \frac{1}{2}; \left(\frac{z}{2 \beta}\right)^2\right)}{(1 - \frac{z}{2})^a} / \]

\{\beta \rightarrow \frac{1}{2}, \alpha \rightarrow -\frac{1}{2}, z \rightarrow e(a, b)^2\} \text{// Simplify}

\[ 4 E\left(1 - \frac{b^2}{a^2}\right) = \sqrt{\frac{2 b^2}{a^2} + 2 \pi \sum_{i=1}^{n} \left(\frac{1}{4}, \frac{1}{4}, 1; \left(\frac{a^2 - b^2}{a^2 + b^2}\right)^2\right)} \]

The hidden symmetry with respect to the interchange \(a \leftrightarrow b\) is revealed.

\[ \text{FullSimplify}[\% , b > a > 0] \]

\[ b E\left(1 - \frac{a^2}{b^2}\right) = a E\left(1 - \frac{b^2}{a^2}\right) \]

Defining

\[ P_4(a, b) = \pi \sqrt{2 (a^2 + b^2)} \sum_{i=1}^{n} \left(\frac{1}{4}, 1; 1; \left(\frac{a^2 - b^2}{a^2 + b^2}\right)^2\right) \]

one can directly check the formula.

\[ \text{Simplify}[P_4(a, b) = P_1(a, b) \text{// FunctionExpand, } a > b > 0] \]

True

\[ \square \text{ Other identities} \]

There are many other possible transformation formulas that can be applied to obtain alternative expressions for the perimeter. For example, using \(\text{functions.wolfram.com/07.23.17.0054.01}\) one obtains the following formula,

\[ P_5(a, b) = P_2(a, b) / \sum_{i=1}^{n} F_1(a, b; c; z) \rightarrow (1 - z)^{a-b+c} \sum_{i=1}^{n} F_1(c - a, c - b; c; z) \]

\[ \frac{16 a^2 b^2 \pi F_1\left(\frac{3}{2}, \frac{3}{2}; 1; 1 - \frac{4 a b}{(a+b)^2}\right)}{(a+b)^3} \]

The perimeter can also be expressed in terms of Legendre functions (see sections 8.13 and 15.4 of [4]). For example, using 15.4.15 of [4] one obtains an elegant and simple symmetric formula

\[ P_6(a, b) = \text{Simplify} \left[ P_2(a, b) / \sum_{i=1}^{n} F_1(a, b; c; x) \right] : \]

\[ \Gamma(a - b + 1) (1 - x)^{-b} (1 + x) \frac{\pi}{2} \sum_{x}^{a-b} \left(\frac{1+x}{1-x}\right)^{c-b} \]

This form can be used to prove that the perimeter of an ellipse is a homogenous mean \((c.f\ [5])\), extending the arithmetic–geometric mean (AGM) already used as a tool for computing elliptic integrals [6].

Using \(\text{functions.wolfram.com/07.07.26.0001.01}\), this gives yet another formula involving complete elliptic integrals.
\[ P_\gamma (a, b) = \]
\[ P_\varepsilon (a, b) / P_\gamma (a, b) \rightarrow 2 F_1 \left( -\nu, \nu + 1; 1; \frac{1 - z}{2} \right) \]

// FunctionExpand // Simplify

\[ 4 \sqrt{ab} \left( 2 E \left( \frac{(a-b)^2}{4ab} \right) - K \left( \frac{(a-b)^2}{4ab} \right) \right) \]

\[ \Box \text{Comparisons} \]

Here we compare the seven formulas obtained above for \( b = 2a \),

\[
\text{Simplify}[\{P_1(a, 2a), P_2(a, 2a), P_3(a, 2a), P_4(a, 2a), P_5(a, 2a), P_6(a, 2a), P_7(a, 2a)\}, a>0]
\]

\[
\{4a E(-3), 3a \pi_2 F_1 \left( \frac{1}{2}, -\frac{1}{2}; 1; \frac{1}{9} \right), \frac{4}{3} a \left( 9 E \left( \frac{1}{9} \right) - 4 K \left( \frac{1}{9} \right) \right) \}
\]

\[
\sqrt{10} a \pi_2 F_1 \left( \frac{1}{4}, -\frac{1}{4}; 1; \frac{9}{25} \right), 64 \pi_2 F_1 \left( \frac{3}{2}, \frac{3}{2}; 1; \frac{1}{9} \right)
\]

\[
2 \sqrt{2} a \pi_1 \left( \frac{5}{4} \right), 4 \sqrt{2} a \left( 2 E \left( -\frac{1}{8} \right) - K \left( -\frac{1}{8} \right) \right)
\]

\[ \text{N[]} \]

\[ \{9.68845 a, 9.68845 a, 9.68845 a, 9.68845 a, 9.68845 a, 9.68845 a, 9.68845 a\]}

\[ \text{Equal} \]

True

and for \( b = a/3 \),

\[
\text{Simplify}[\{P_1(a, a/3), P_2(a, a/3), P_3(a, a/3), P_4(a, a/3), P_5(a, a/3)\}, a>0]
\]

\[
\{4a E \left( \frac{8}{9} \right), 4 a \pi_2 F_1 \left( -\frac{1}{2}, -\frac{1}{2}; 1; \frac{1}{4} \right), 2 a \left( 8 E \left( -\frac{1}{4}, -\frac{1}{4}; 1; \frac{16}{25} \right) \right) \}
\]

\[
2 a \pi_2 F_1 \left( \frac{3}{2}, \frac{3}{2}; 1; \frac{1}{4} \right), 2 a \pi_1 \left( \frac{5}{4} \right), 2 a \pi_1 \left( -\frac{1}{4}; -\frac{1}{4}; 1; \frac{16}{25} \right)
\]

\[ \text{N[]} \]

\[ \{4.45496 a, 4.45496 a, 4.45496 a, 4.45496 a, 4.45496 a, 4.45496 a, 4.45496 a\]}

\[ \text{Equal} \]

True

\[ \text{\underline{Numerical Approximation}} \]

At \( \text{www.ebyte.it/library/docs/math05a/EllipsePerimeterApprox05.html} \) \[1\] one is encouraged to search for "an efficient formula using only the four algebraic operations (if possible, avoiding even square-root) with a maximum error below 10 parts per million. If would be also nice if such a formula were exact for both the circle and the degenerate flat ellipse."

The Gauss–Kummer series expressed as a function of the homogenous variable
\[ h = 1 - 4ab/(a+b)^2 \], reads
\[
\text{GaussKummer}[h_] = \frac{P_2(a, b)}{a + b} / (a + b) \rightarrow 2 \sqrt{a} b / \sqrt{1 - h}
\]

\[
\pi_2 F_1 \left( -\frac{1}{2}, -\frac{1}{2}; 1; h \right)
\]

## Series expansions

The series expansion about \( h = 0 \) is useful for small \( h \).

\[
\text{GaussKummer}[h] + O[h]^9
\]

\[
\pi + \frac{\pi h}{4} + \frac{\pi h^2}{64} + \frac{\pi h^3}{256} + \frac{25 \pi h^4}{16384} + \frac{49 \pi h^5}{65536} + \frac{441 \pi h^6}{1048576} + \frac{1089 \pi h^7}{4194304} + \frac{184041 \pi h^8}{1073741824} + O(h^9)
\]

Around \( h = 1 \), terms in \( \log(1-h) \) arise.

\[
\text{Simplify}[\text{Series}[\text{GaussKummer}[h], \{h, 1, 2\}, 0 < h < 1]]
\]

\[
4 + (h - 1) + \frac{1}{16} \left( -2 \log(1 - h) - 4 \psi^{(0)} \left( \frac{3}{2} \right) - 4 \gamma + 3 \right) (h - 1)^2 + O(h - 1)^3
\]

Using functions.wolfram.com/07.23.06.0015.01 we can obtain the general term of this series (c.f. 17.3.33–17.3.36 of [4]),

\[
\text{GaussKummer}[h] / . \_ \_ F_1 (a, b; c; z) \rightarrow \text{With}[\{n = c - a - b\},
\]

\[
\frac{\Gamma(a + b + n)}{\Gamma(a) \Gamma(b)} \left( \sum_{k=0}^{\infty} \frac{(a + n)_k (b + n)_k}{k! (k + n)!} (-log(1-z) + \psi(k + 1) +
\]

\[
\psi(k + n + 1) - \psi(a + k + n) - \psi(b + k + n) (1 - z)^k
\]

\[
\right)(z - 1)^n +
\]

\[
\frac{(n - 1)! \Gamma(a + b + n)}{\Gamma(a + n) \Gamma(b + n)} \sum_{k=0}^{n-1} \frac{(a)_k (b)_k (1 - z)_k}{k! (1 - n)_k} \] // Simplify

\[
\frac{1}{4} \left( \sum_{k=0}^{\infty} \frac{(1 - h)_k \left( \frac{3}{2} \right)_k^2}{k!} \right) \left( -log(1 - h) + \psi^{(0)}(k + 1) - 2 \psi^{(0)}(k + \frac{3}{2}) + \psi^{(0)}(k + 3) \right)
\]

\[
(h - 1)^2 + 4 (h + 3)
\]

## Polynomial Approximants

**Linear Approximant**

From the exact values at \( h = 0 \),

\[
\text{GaussKummer}[0] = \pi
\]

and at \( h = 1 \),

\[
\text{GaussKummer}[1] = 4
\]
one constructs the linear _extreme perfect_ approximant.

\[
\text{Linear}[h_] = (1 - h) \text{GaussKummer}[0] + h \text{GaussKummer}[1] \text{ // Simplify} \\
\pi - h(-4 + \pi)
\]

\[
\text{Plot}[[\text{GaussKummer}[h], \text{Linear}[h]], \{h, 0, 1\}]
\]

**Quadratic Approximant**

The quadratic approximant, exact at \(h = 0, 1/2, 1\),

\[
\text{Table}[[h, \text{GaussKummer}[h]], \{h, 0, 1, \frac{1}{2}\}] \text{ // FullSimplify} \\
\begin{pmatrix}
0 & \pi \\
\frac{1}{2} & \frac{\sqrt{\pi} \Gamma(\frac{1}{2})}{2} + \frac{\Gamma(\frac{1}{2}) \Gamma(\frac{3}{2})}{\sqrt{\pi}} \\
1 & \frac{3.14159}{4}
\end{pmatrix}
\]

\[
\text{Quadratic}[h_] = \text{InterpolatingPolynomial}[\% , h] \text{ // N} \\
(0.0891819 (h - 0.5) + 0.813816) h + 3.14159
\]

has a maximum absolute relative error of \(\leq 8 \times 10^{-4}\).
\begin{itemize}
  \item Graphics
\end{itemize}

**\textbf{\textit{n}}^{\text{th}} \textit{order polynomial Approximant}**

Here is the \(n\)\textsuperscript{th}–order “even–tempered” polynomial approximant, exact at \(h = m \div n\) for \(m = 0, 1, \ldots, n\).

\[
poly[n_] := poly[n] = \text{Function}[h, \text{Evaluate}@\text{InterpolatingPolynomial}[N@\text{Table}[[h, GaussKummer[h]], \{h, 0, 1, \frac{1}{n}\}], h]]
\]

The \(9\)\textsuperscript{th}–order approximant has a maximum absolute relative error of \(< 10 \times 10^{-6}\).

\[
\text{Plot}[10^6 \left(1 - \frac{\text{poly}[9][h]}{\text{GaussKummer}[h]}\right) \{h, 0, 1\}, \text{PlotRange} \to \text{All}, \text{PlotPoints} \to 30]
\]

\begin{itemize}
  \item Graphics
\end{itemize}

**\textbf{\textit{Chebyshev polynomial Approximant}}**

Sampling the Gauss–Kummer function at the zeros of \(T_n(2x - 1)\), which are at \(x_m = \cos^2((m + 1/4) \frac{\pi}{n})\), yields a Chebyshev polynomial approximant.
ChebyshevPoly[n_] :=
    ChebyshevPoly[n] = Function[h, Evaluate[InterpolatingPolynomial[
        Join[{{0, GaussKummer[0]}, {1, GaussKummer[1]}},
            Table[{Cos[(m + 1/4) π/n], GaussKummer[Cos[(m + 1/4) π/n]]},
                {m, n}]], h]]

The 8th–order approximant has a maximum absolute relative error of $\approx 7 \times 10^{-6}$.

Plot[10^6 \left(1 - \frac{\text{ChebyshevPoly}[8][h]}{\text{GaussKummer}[h]}\right), \{h, 0, 1\}, PlotRange -> All]

• Graphics •

**Rational Approximation**

After loading the package (stub),

<< NumericalMath`

eone obtains a family of $[N, M]$ rational polynomial minimax approximations.

GKApprox[n_, m_] := GKApprox[n, m] = Function[h, Evaluate[
    MiniMaxApproximation[GaussKummer[h], \{h, \{0, 1\}, n, m\}][2, 1]]

For example, the [4,3] minimax approximation,

GKApprox[4, 3][h]

\[-0.0811183 h^4 + 0.273498 h^3 + 1.77163 h^2 - 5.0554 h + 3.14159\]
\[-0.14146 h^3 + 1.01321 h^2 - 1.8592 h + 1\]

has (absolute) relative error $\approx 2.3 \times 10^{-7}$, but is not "extreme perfect".
Using the linear approximant, \( 4h + \pi(1-h) \), and noting that \( h(1-h) \) vanishes at both \( h=0 \) and \( h=1 \), leads to an optimal \([N+2, M]\) extreme perfect approximant of the form

\[
\pi \, _2F_1 \left( -\frac{1}{2}, -\frac{1}{2}; 1; h \right) \approx 4h + \pi(1-h) + \alpha h(1-h) \sum_{i=1}^{N} \frac{(h-p_i)}{\prod_{j=1}^{M} (h-q_j)},
\]

where the parameters \( \alpha, \{p_i\}_{i=1}^{N}, \) and \( \{q_j\}_{j=1}^{M} \) need to be determined.

Implementation of the approximant is immediate.

\[
\text{EllipseApproximant}[\alpha,\{p\}_{\text{List}},\{q\}_{\text{List}}] :=
\text{Function}[h, \text{Evaluate}[4h + \pi(1-h) + \alpha h(1-h) \frac{\text{Times} @@ (h-p)}{\text{Times} @@ (h-q)}]]
\]

After uniformly sampling the Gauss-Kummer function,

\[
\{\text{xdata}, \text{ydata}\} = \text{Table}[[h, \text{GaussKummer}[h]], \{h, 0, 1, 0.001\}] // \text{Transpose};
\]

one can use \textit{NMinimize} and the \( \infty \)-norm to obtain the accurate approximants. For example, the (almost) optimal \([3, 2]\) approximant is computed using

\[
\text{NMinimize} \left[ \|\text{ydata} - \text{EllipseApproximant}[\alpha, \{p\}, \{q, r\}]\|_{\infty}, \begin{pmatrix} \alpha & 0.22 & 0.24 \\ p & 1.25 & 1.35 \\ q & 3.4 & 3.5 \\ r & 1.15 & 1.25 \end{pmatrix} \right]
\]

\[
(0.0000140975, \{p \to 1.28546, q \to 3.475, r \to 1.19671, \alpha \to 0.235456\})
\]

leading to

\[
\text{EllipseApproximant}[\alpha, \{p\}, \{q, r\}][h] /. \text{Last}[%]
\]

\[
0.235456 (h - 1.28546) h (1-h) \frac{\pi (1-h) + 4h}{(h-3.475)(h-1.19671)}
\]

This simple approximant has (absolute) relative error \( \leq 4 \times 10^{-6} \).
Conclusions

*Mathematica* is an ideal tool for developing accurate approximants to special functions because:

- all special functions of mathematical physics are built–in and can be evaluated to arbitrary precision for general complex parameters and variables;
- standard analytical methods—such as symbolic integration, summation, series and asymptotic expansions, and polynomial interpolation—are available;
- properties of special functions—such as identities and transformations—are available at MathWorld [6] and the Wolfram functions Site [7] and, because these properties are expressed in *Mathematica* syntax, can be used directly;
- relevant built–in numerical methods include rational polynomial approximants, minimax methods, and numerical optimization for arbitrary norms;
- visualization of approximants can be used to estimate the quality of approximants; and
- combining these approaches is straightforward and leads, in a natural way, to optimal approximants.

This paper uses the exercise of computing the perimeter of an ellipse using a simple set of approximants to illustrate these points.

References

[1] [http://www.ebyte.it/library/docs/math05a/EllipsePerimeterApprox05.html](http://www.ebyte.it/library/docs/math05a/EllipsePerimeterApprox05.html)


[3] [http://www.alumni.iitb.ac.in/mathslecture.htm](http://www.alumni.iitb.ac.in/mathslecture.htm)
[4] Abramowitz and Stegun online:
http://www.convertit.com/Go/ConvertIt/Reference/AMS55.ASP

